amU
Aix Marseille Université

NNT : 2024AIXM0001

THÈSE DE DOCTORAT

Soutenue à AMU — Aix-Marseille Université
le 7 octobre 2026 par

Yann-Situ GAZULL

On the Computations of Homological Holes

where topology, geometry and combinatorics meet

Discipline
Informatique

École doctorale
ED 184 - Mathématiques et Informatique

Laboratoire
Laboratoire d'Informatique et des Systèmes
(LIS, UMR 7020)

Composition du jury

Steve OUDOT Directeur de recherche, INRIA, Saclay	Rapporteur
María José JIMÉNEZ Associate professor, Universidad de Sevilla	Rapporteure
Guillaume DAMIAND Directeur de recherche, CNRS, Lyon	Président du jury
Emmanuel GODARD Professeur des universités, AMU	Examinateur
Samuel PELTIER Professeur des universités, Université de Poitiers	Examinateur
Julien TIERNY Directeur de recherche, CNRS, Sorbonne	Examinateur
Alexandra BAC Maîtresse de conférence, AMU	Directrice de thèse
Aldo GONZALEZ-LORENZO Maître de conférence, AMU	Co-encadrant de thèse

LIS LABORATOIRE D'INFORMATIQUE & DES SYSTÈMES

Affidavit

I, undersigned, Yann-Situ GAZULL, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific supervision of Alexandra BAC and Aldo GONZALEZ-LORENZO, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the french national charter for Research Integrity and AMU charter on the fight against plagiarism.

This work has not been submitted previously either in this country or in another country in the same or in a similar version to any other examination body.

Marseille, June 15 2026





This work is licensed under Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International Public License

Liste de publications et participation aux conférences

Liste des publications réalisées dans le cadre du projet de thèse:

1. **Yann-Situ GAZULL**, Alexandra BAC and Aldo GONZALEZ-LORENZO. "Computing Geometrical Measures of Topological Holes". In: *Computer-Aided Design 163* (2023). DOI: https://doi.org/10.1016/j.cad.2023.103563.
2. Aldo GONZALEZ-LORENZO, Alexandra BAC and **Yann-Situ GAZULL**. "A constructive Approach of Alexander duality". In: *Journal of Applied and Computational Topology 9.2* (Dec. 2024). DOI: https://doi.org/10.1007/s41468-024-00198-1.
3. **Yann-Situ GAZULL**, Aldo GONZALEZ-LORENZO and Alexandra BAC. "Characterization of the Computed Homology and Cohomology bases". In: *International Conference on Discrete Geometry and Mathematical Morphology* (Nov. 2025). DOI: https://dx.doi.org/10.1007/978-3-032-09544-2_12.
4. **Yann-Situ GAZULL**, Aldo GONZALEZ-LORENZO and Alexandra BAC. "Characterization of Homology and Cohomology bases Computed by Standard Methods". The extended and comprehensive version of the previous publication. In: DGMM25 special issue, *Journal of Mathematical Imaging and Vision* (2026), **to be published**.
5. **Yann-Situ GAZULL**, Aldo GONZALEZ-LORENZO and Alexandra BAC. "On the Graph of Homology Computations and its Connectedness", (2025) **preprint**. HAL: https://hal.science/hal-05763373.

Publication réalisée en parallèle du projet de thèse

6. **Yann-Situ GAZULL**. "Stability and extension of steady and ranging persistence". In: *Journal of Applied and Computational Topology 10.1* (Dec. 2025). DOI: http://dx.doi.org/10.1007/s41468-025-00228-6.

Participation aux conférences et écoles d'été au cours de la période de thèse:

Présentations dans le cadre du doctorat, lors de conférences, workshops et Groupes de travail

1. *R-GTMG*. Marseille (CIRM). March 2024.
   Presentation: *A Constructive Approach to Alexander Duality*.
2. *Geometry and Computing*. Marseille (CIRM). October 2024.
   Presentation: *A Versatile Tool for Computing and Understanding Homology*.
3. *GTMG 2025*. Poitiers. March 2025.
   Presentation: *Ongoing Works about Homology Configurations*, best presentation prize.
4. *CGAL Days*. Sophia Antipolis. April 2025.
   Presentation: *Homological Discrete Vector Fields... 1 year later...*, presented with Alexandra BAC.
5. *Séminaires Topocs*. Marseille. May, June and November 2025.
   Presentations: *Structures in Combinatorial Homology*, *A Constructive Approach to Alexander Duality*, *Homology Configurations*.
6. *DGMM 2025*. Gröningen, Nederland. November 2025.
   Presentation: *Characterization of the Computed Homology and Cohomology bases*.
7. *GTMG 2026*. Paris (Jussieu). March 2026.
   Presentation: *Experiments on the Space of Homology Computations*, presented with Menjatiana ANDRIANABINITSOA.

Présentations dans le cadre de stages pré-doctoraux

1. *GTMG 2021*. Online. March 2021.
   Presentation: *Measuring Holes of 3D Meshes*, best student presentation prize.
2. *Demi-journée du Pôle Calcul*. Marseille. December 2022.
   Presentation: *Étude d'Objets Homologiques Computationnels*.
3. *SPM 2023*. Genova, Italia. July 2023.
   Presentation: *Computing Geometrical Measures of Topological Holes*, second best paper award.

Autres participations

1. *J-FIG 2023*. Montpellier. November 2024.
2. *Mois thématique*. Marseille (CIRM). February 2024.
3. *J-FIG 2024*. Strasbourg. October 2024.
4. *Climathics*. Marseille (CIRM). January 2025.
5. *Illustration as a Mathematical Research Technique*. Marseille (CIRM) and Paris (IHP). January - February 2026.
6. *20 ans du GDR-IFM*. Paris (Jussieu). March 2026.
7. *Mathematic summer camp*. Nesin Matematik Köyü (Turkey). August 2026.

Abstract and keywords

On the Computations of Homological Holes

where topology, geometry and combinatorics meet

Homology is a branch of algebraic topology that intuitively captures information about the holes of an object. This thesis is dedicated to the study of homology computations from both combinatorial and geometric perspectives. Our work is centered on *homological discrete vector fields* (HDVFs), a combinatorial structure that computes homological information and encompasses several approaches, including discrete Morse theory, standard persistent homology, and tri-partitions.

We first investigate the combinatorial properties of HDVFs. In particular, we characterize the homology and cohomology bases computed by HDVFs and related methods. We then adopt a global perspective, refining the connection between filtrations, persistent homology, and HDVFs, and proving that the graph of perfect HDVFs is connected.

We further explore the interaction between homology and combinatorics by developing a constructive version of Alexander duality. To study the resulting constructions, we introduce *homology configurations*, which classify HDVFs and provide a higher-level point of view on homology computations. In particular, we show that homology configurations define a topological invariant for objects of dimension at most 3.

Finally, we shift our focus to the geometry of homological holes. We introduce the *thickness* and *breadth* of a homological hole, which quantify complementary aspects of its geometry. We propose theoretical methods for computing these measures, implement practical discretizations, and present results on three-dimensional objects.

Overall, this thesis demonstrates that HDVFs provide a relevant framework for studying the combinatorics of homology computations. More generally, it investigates homology computations from different global perspectives and introduces new representations of homological holes.

Keywords: computational homology, combinatorial topology, algebraic topology, persistent homology, discrete geometry, geometric modeling.

Résumé et mots clés

Sur les Calculs des Trous Homologiques
lorsque topologie, géométrie et combinatoire se rencontrent

L'homologie est une branche de la topologie algébrique qui permet de capturer intuitivement les informations liés aux trous d'un objet. Cette thèse est dédiée à l'étude des calculs d'homologie sous un angle à la fois combinatoire et géométrique. Nos travaux sont centrés sur les *champs de vecteurs discrets homologiques* (HDVF), une structure combinatoire qui calcule des informations homologique et englobe plusieurs approches, notamment la théorie de Morse discrète, l'homologie persistante standard et les tri-partitions.

Nous étudions d'abord les propriétés combinatoires des HDVFs. En particulier, nous caractérisons les bases d'homologie et de cohomologie calculées par les HDVFs et les méthodes associées. Nous adoptons ensuite un point de vue globale, en affinant le lien entre filtrations, homologie persistante et HDVFs, et en démontrant que le graphe des HDVFs parfaits est connexe.

Nous explorons ensuite plus profondément l'interaction entre homologie et combinatoire, en développant une version constructive de la dualité d'Alexander. Pour étudier les constructions qui en résultent, nous introduisons les *configurations d'homologie*, qui classifient les HDVFs et offrent un point de vue de plus haut niveau sur les calculs d'homologie. En particulier, nous montrons que les configurations d'homologie définissent un invariant topologique pour les objets de dimension au plus égale à 3.

Enfin, nous déplaçons l'attention sur la géométrie des trous homologiques. Nous introduisons l'*épaisseur* et la *largeur* d'un trou homologique, qui quantifient des aspects complémentaires de sa géométrie. Nous proposons des méthodes théoriques pour calculer ces mesures, implémentons des discrétisations pratiques et présentons des résultats sur des objets tridimensionnels.

Dans l'ensemble, cette thèse démontre que les HDVFs constituent un cadre pertinent pour l'étude de la combinatoire des calculs d'homologie. Plus généralement, elle explore les calculs d'homologie sous différents angles et introduit de nouvelles représentations des trous homologiques.

Mots clés: homologie computationelle, topologie combinatoire, topologie algébrique, homologie persistante, géométrie discrète, modélisation géométrique.

Remerciements

Je tiens à remercier l'ensemble des personnes ayant influencé, de près ou de loin, directement ou indirectement, ces travaux de recherche.

Mes premières pensées vont naturellement à mes merveilleux encadrants. Merci Alexandra pour ton enthousiasme légendaire, qui donne de la motivation peu importe le contexte. Merci pour ton dévouement, à travailler tard et tôt pour m'accorder du temps malgré les douze travaux avec lesquels tu jongles parfois. Merci Aldo pour nos sessions arlésiennes, qui me faisaient une belle parenthèse, après lesquelles je repartais en train parfois avec une réponse, toujours avec de nouvelles questions trépidantes. Merci pour ton écoute, tes conseils, tes super idées et ton thé au sirop d'érable. Il y a plus de 5 ans, vous m'expliquiez ce qu'était un trou selon l'homologie. Depuis, j'ai sincèrement aimé réfléchir devant le tableau, debugger au Café des thés, discuter de trou et de rien avec vous. J'ose dire que les années de thèse ont été un vrai plaisir à vos côtés.

Je suis reconnaissant envers les membres de mon jury de thèse, qui ont accepté de prendre du temps pour s'intéresser à mes travaux, en particulier Steve Oudot et Maria-José Jimenez, qui ont accepté de lire ce gros bébé de presque 230 pages, avec ses preuves techniques parfois indigestes. Merci beaucoup pour vos suggestions. Je remercie aussi Samuel Peltier, Emmanuel Godard et Patrizio Frosini pour leur suivi au sein du CSI.

Je remercie l'équipe GMOD, les anciens et les nouveaux, les luminiens et les arlésiens, les permanents et non-permanents. Pour ne citer que les plus jeunes : Yann, Clémence, Samantha, Yiyun, Grekou, Gregoire, Tibault, Florian et merci Mathieu pour tes géniales réflexions sur l'univers et les transports marseillais. Merci le reste du LIS, notamment les CANA, les IM, Mohssine, Aliénor, Cameron, Yannis, David, le séminaire Topocs, et les autres. Obrigado Emmanuel pelas informações sobre os grafos e o Brasil. Merci Ivan pour ta passion mathématique, tes conseils et tes propositions. Merci aux géniaux stagiaires qui ont participé à ces travaux de recherche : Danih, Madhav. Merci aux matheux et matheuses de l'I2M pour vos séminaires doctorants. Merci Polytech, Luminy, amU, les poufs et alvéoles de l'hexagone, le CROUS.

Un peu plus loin qu'à Marseille, merci Étienne pour ton intérêt, merci les membres du GTMG, e grazie Massimo Ferri per il tuo prezioso aiuto.

Au delà du milieu académique, mes pensées vont à toutes les personnes ayant accompagné mon quotidien durant ces années marseillaises.

Les personnes vivant ou ayant vécu avec moi dans ce chouette 2-trou avec jardin : Yuka, Jana, Iulia, a daily pleasure to live with you < 3 ; merci aussi Bérénice, Aliénor,

Joni, Paul, Élodie et Alex, ainsi que Patate, Boubou, et même Nola.

La communauté marseillaise de go (jeu passionnant où deux 1-trous procurent l'immortalité) au sein de laquelle j'ai trouvé de belles amitiés : Alex, Alexandra, Anton, Axel, Camilo, Florian, Jonathan, Laura (je te dois toujours une partie avec des baklavas), Morgan, Morgane, Nicolas, Paul-Gau, Pierre, Tadj, Théo, Thomas et tous les autres spécimens de la Sardin Ki'in, ainsi que les personnes gravitant autour du Nain Jaune comme Logan, Kyle, Nico, les vampires, et Philémon qui je l'espère repose en paix.

Les personnes avec qui j'aime pratiquer l'impro, discipline où le trou de mémoire n'est pas bien défini : cette chouette famille des Bullos, Alex, Caro, Cécile, Célia, Emy, Loïc, Maëva, Marlène, Océ, Pierre, Raf, Seirigne. Merci Aurélien et Théo pour vos stages passionnants qui m'ont remis dans le bain.

Les Muleks, avec qui je joue des morceaux de musique à déformation continue près : la liste est immense, merci à vous toutes et tous, vous m'avez apporté beaucoup de joie et d'énergie ! Citons simplement mes super compagnons d'apprentissage du timbal Adlane, Alice, Felipe, Tatiana, et aussi Jade.

Les jongleurs d'Extrême Jonglerie, qui dessinent des topologies folles avec les trajectoires des massues : merci Nico et Paul pour m'avoir tant appris sur le passing.

Et tous les marseillais et marseillaises restants: Juju-sans-chichi pour les plans Cévennes et les plans dub à perpet' ; Thym, avec qui j'adore parler maths, jeux, vie, et autres, j'espère que tu vas te plaire en IMD ; l'autre Thym, grand sorcier de la percussion, avec qui j'adore aussi parler de maths ; Jade, alter ego que je ne vois que trop rarement, faut qu'on fasse un truc ensemble, n'importe quoi! Les fusionnelles Milène et Héla, l'impressionnant Tom aka "le MacGyver qui court en tongue", Zoë l'artiste championne vertigineuse, Clarux l'inconnue trop sympa du train, Angel aux réflexions inspirées et inspirantes, Gwendo qui jamais ne met un faux-plan, Clara dont j'admire la sincérité et la gentillesse, Flo et sa niaque associative, Marin le légendaire poète des Calanques, et les autres luminiens de 2021. Alexis et Léandre, deux piliers à Marseille, merci pour vos amitiés solides. Mohssine, Sabri, Roxanne, Sandro, la fine équipe. Sans oublier Kaki, Mathilde, Maud, Lucas, Antoine, Magali, Yaëlle, Alé, Adou, Clément, Titi.

Pour remonter avant Marseille, je remercie les proches des différentes périodes de ma vie, du Gard à L'Émilie-Romagne, en passant par le Rhône. Merci les ami(e)s de Bologna, de vrais coups de cœur. Les ami(e)s de Minvayu à Auroville. Les ami(e)s de la promo bip-boup de l'ENS de Lyon : Eliot le pote bien-sûr, mais aussi Mathieu, Alexandre, Gaëtan, Nina, Loïc, Orégane, que j'ai pu recroiser par-ci par-là. Les autres ami(e)s de l'ENS : entre autres AB, Ramion, Hélène, Alfred, Antoine, Lucas, Arthur, Lætitia. Mes deux précieux vestiges de prépa, Mario et Damien. Nono, un noyau, ainsi que les autres ami(e)s du 7ème (et autres étages) du CROUS de Puvis de Chavanne, c'est pour la vie apparemment. Les amis de l'internat et du lycée, que je vois très rarement, mais à qui je pense régulièrement. Merci notamment à toi, Félix, de m'avoir mis à la programmation. Merci Nicolas pour ta bonne bouffe et ta longue amitié, ça commence à dater les chevaliers du Vignet... Merci Tiphaine pour ta précieuse et

régulière présence, et pour tes histoires incroyables à tomber parterre.

Des mercis infinis à vous, mes parents, pour votre soutien éternel. Akpe na wò kaka, enɔ̃ nye lɔ̃lɔ̃tɔ. Merci les tatas et tontons d'avoir suivi mon parcours de près. Merci Popol, le premier témoin de ma fascination pour les hypercubes, qui aurait pensé que ça m'emmènerait si loin ? Merci Nadège pour ton attention, je suis fan et fier de toi.

Je dois aussi beaucoup à la rubrique *magic maths* de science et vie junior, au *Petit démon des maths* d'Enzensberger, à M.Ramangasalama, à *Choux romanesco Vache qui rit et intégrales curvilignes* l'incroyable blog d'eljj, à J.H.Conway, et surtout à M.Terraza, pour avoir suscité ma passion pour les mathématiques.

Pour finir, mes pensées vont à Laveur-de-pont Sandro, mon co-moussaillons à bord du Funky-Poulet, qui égaye si facilement la vie de ses milles bêtises, de sa bonhomie, de son groove et de son énergie mathématique tout droit puisée dans sa foi en god-Thurston. Merci pour ton amitié et ton soutien indéfectible.

Enfin, Charlie, la vie avec toi c'est encore mieux que de jouer aux maths. Merci... pour tout, je crois bien.

Contents

Liste de publications et participation aux conférences 3

Abstract and keywords 5

Résumé et mots clés 6

Remerciements 7

Contents 10

Introduction (Français) 13

Introduction (English) 17

1. Common Background 21
- 1.1. An Introductory Example . . . 22
- 1.2. Geometric Complexes: Discretizations of Topological Spaces . . . 25
  - 1.2.1. Simplicial complexes . . . 25
  - 1.2.2. CW-complexes . . . 26
  - 1.2.3. Other geometric complex constructions . . . 27
- 1.3. Chain Complexes and Homology . . . 27
  - 1.3.1. General chain complexes . . . 28
  - 1.3.2. Homology groups . . . 29
  - 1.3.3. Singular chain complexes . . . 31
  - 1.3.4. Introducing combinatorics: combinatorial chain complexes . . . 32
  - 1.3.5. Geometric chain complexes . . . 34
  - 1.3.6. Intuition, holes and homology bases . . . 37
  - 1.3.7. Cohomology . . . 38
  - 1.3.8. A word about homotopy . . . 40
- 1.4. Computing Homology . . . 40
  - 1.4.1. The Smith normal form . . . 42
  - 1.4.2. Reductions and effective homology . . . 43
  - 1.4.3. Discrete Morse theory and DGVFs . . . 44
- 1.5. Persistent Homology . . . 45
  - 1.5.1. Definition . . . 46
  - 1.5.2. Stability . . . 47
  - 1.5.3. Computing persistent homology . . . 47
  - 1.5.4. Extensions of persistent homology . . . 48

1.6. Homological Discrete Vector Fields 49
1.6.1. Definition and main properties 50
1.6.2. Elementary swapping operations 56
1.6.3. Completion of HDVF 60
1.6.4. Emulating standard computational homology methods 62

2. Combinatorial Properties of HDVFs **64**
2.1. Introduction 64
2.2. Tri-Partitions 65
2.2.1. Definition 65
2.2.2. Relation between Tri-partitions and perfect HDVFs 66
2.2.3. The canonical (co)cycle lemma 68
2.3. (Co)Homology Bases Computed by HDVF and Related Methods 69
2.3.1. Explicit homology bases 70
2.3.2. Characterization of the homology bases computed by HDVFs . 73
2.3.3. Explicit cohomology bases 78
2.3.4. Combining explicit homology and cohomology bases 80
2.3.5. Characterization of homology bases computed by other methods 83
2.4. Conclusion 84

3. The Space of HDVFs and Homology Computations **86**
3.1. Introduction 86
3.2. Persistent Homology and HDVFs 87
3.2.1. The standard abstract persistence algorithm 87
3.2.2. Emulating persistent homology with HDVF 88
3.2.3. Persistent generators 93
3.2.4. About the filtration space 94
3.3. The Graph of HDVFs 97
3.3.1. The graph of perfect HDVFs 97
3.3.2. The connectedness algorithm 98
3.3.3. Proof of the connectedness 100
3.3.4. Experimental results on the graph of HDVFs 104
3.4. Adapting a Perfect HDVF to a Given Explicit Basis 107
3.4.1. The adaptation algorithm 108
3.4.2. Proof of the correctness of the adaptation 110
3.5. Conclusion 113

4. Beyond Standard Homology Information **115**
4.1. Introduction 115
4.2. A Constructive Approach to Alexander Duality 117
4.2.1. Relative and reduced homology 117
4.2.2. Combining complexes and HDVFs 119
4.2.3. The combinatorial duality 123
4.2.4. Combinatorial Alexander duality for geometric complexes . . . 127
4.2.5. Singular Alexander duality for simplicial complexes 130

4.2.6. Homology quartets . . . 139
4.3. Homology Configurations . . . 142
4.3.1. Motivations . . . 142
4.3.2. Definition and intuition . . . 143
4.3.3. Properties . . . 145
4.3.4. Combinatorial invariance . . . 150
4.3.5. Topological invariance and the Hauptvermutung . . . 162
4.3.6. Computing homology configurations . . . 164
4.4. About the Constructiveness of Homology Morphisms . . . 169
4.4.1. Constructible homology-cohomology isomorphisms . . . 169
4.4.2. Constructible combinatorial duality isomorphisms . . . 170
4.5. Conclusion . . . 177

5. Geometric Measures for Homological Holes 179
5.1. Introduction . . . 179
5.2. Hole-Balls . . . 181
5.2.1. The signed-distance filtration . . . 181
5.2.2. Thickness, breadth and TB-pairs . . . 182
5.3. The Medial Axes . . . 183
5.3.1. Definition . . . 184
5.3.2. Geometrical and topological properties . . . 184
5.4. Computing Hole-Balls from the Medial Axes . . . 188
5.4.1. Overview of our approaches . . . 188
5.4.2. Link approach . . . 189
5.4.3. Medial axes approach . . . 190
5.5. Implementation and Validation on 3D Objects . . . 196
5.5.1. Discretization of our approaches . . . 196
5.5.2. Implementation and results . . . 198
5.5.3. Non-present hole-balls also gives information . . . 200
5.5.4. Limitation: obtaining an appropriate sampling . . . 200
5.5.5. Catching the right ball centers . . . 202
5.5.6. Complexity . . . 203
5.6. Prospects: Defining and Computing Hole Shapes . . . 204
5.7. Conclusion . . . 207

Conclusion 209

Bibliography 212

ANNEXES 223

A. Supplementary proofs 224

B. Gallery 228

Introduction (Français)

La topologie étudie les propriétés des formes, considérées comme des espaces, qui sont préservées par déformations continues. Parmi elles figurent la connexité, l'adjacence, les lacets et les trous. L'un des objectifs centraux de la topologie est de déterminer si deux espaces sont équivalents à déformation continue près. Deux tels espaces sont dits *homéomorphes*.

Pour répondre à cette question, les mathématiciens ont introduit des *invariants topologiques*. Ce sont des objets mathématiques définis sur les espaces, qui demeurent équivalents sur des espaces homéomorphes. Parmi ces invariants, l'*homologie*, personnage principal de ce manuscrit, capture intuitivement l'information liée aux trous d'un espace. Une intuition fondamentale est que deux espaces possédant un nombre de trous différents ne sont pas homéomorphes.

L'interaction entre combinatoire et topologie apparaît lorsqu'un espace est discrétisé, c'est-à-dire décomposé en cellules élémentaires. Étudier la topologie à travers des discrétisations amène des idées fructueuses. Un exemple classique est la caractéristique d'Euler, qui capture de l'information topologique en comptant simplement les cellules d'une discrétisation.

D'un point de vue historique, la topologie algébrique était désigné sous le nom de *topologie combinatoire* durant la première moitié du 20ème siècle. Ce nom reflétait l'approche courante consistant à étudier les variétés à travers leurs triangulations. Plus tard, au milieu du siècle, la discipline s'est orientée vers l'algèbre, suivant le développement de nouveaux outils algébriques tels que la théorie des catégories. Plus récemment, les interactions entre combinatoire et topologie ont donné naissance à l'homologie computationnelle, qui produit aujourd'hui de nombreuses applications en analyse topologique des données.

Cette thèse explore l'homologie et ses calculs, principalement d'un point de vue combinatoire, mais également géométrique. Plus largement, elle étudie différentes interactions entre topologie, combinatoire et géométrie.

Une part importante de cette thèse est consacrée à l'interaction entre combinatoire et homologie, à travers le prisme des *champs de vecteurs discrets homologiques* (HDVF). Les HDVFs sont des objets hybrides à l'interface entre combinatoire et algèbre, qui encodent de l'information homologique. Ils sont apparus comme une généralisation de la théorie de Morse discrète, enrichie d'idées issues de l'homologie effective. Tout au long de cette thèse, les HDVFs sont à la fois considérés comme objets d'étude et comme outils théoriques pour l'analyse des calculs d'homologie. Nous résumons brièvement le contenu des chapitres, voir la Figure 1 pour une illustration.

Chapitre 1. Contexte et Notions Préliminaires

Le Chapitre 1 introduit les notions préliminaires et le contexte scientifique de cette thèse. Nous commençons par un exemple introductif. Nous introduisons les principales discrétisations des espaces topologiques, la théorie de l'homologie et les complexes de chaînes associés. Nous présentons ensuite différentes approches de calcul homologique, ainsi que l'homologie persistante. Nous concluons par une introduction aux HDVFs.

Chapitre 2. Propriétés Combinatoires des HDVFs

Le Chapitre 2 (première contribution) étudie les propriétés des HDVFs sous un angle « local ». Nous montrons tout d'abord l'équivalence entre les *tri-partitions*, introduites par H. EDELSBRUNNER and ÖLSBÖCK 2020, et les HDVFs parfaits. Cette équivalence apporte une meilleure compréhension de ces deux concepts en tant que structures combinatoires.

Dans un second temps, nous caractérisons les bases d'homologie et de cohomologie calculées par les HDVFs. Précisément, nous introduisons les bases de (co)homologie *explicites*, combinant à la fois combinatoire et algébre et généralisant les bases strictement fondamentales en théorie des graphes. Nous montrons qu'elles correspondent exactement aux bases d'homologie calculées par les HDVFs. Cette caractérisation se transfère naturellement aux méthodes similaires.

Enfin, nous établissons une condition nécessaire et suffisante pour qu'une paire de bases explicites d'homologie et de cohomologie soit induite par un unique HDVF.

Chapitre 3. L'Espace des HDVFs et des Calculs Homologiques

Le Chapitre 3 (deuxième contribution) adopte une perspective plus « globale » sur les HDVFs et les calculs d'homologie. Nous commençons par approfondir la relation entre homologie persistante et HDVF, en montrant qu'une filtration induit un unique HDVF, et que tout HDVF est induit par au moins une filtration.

Nous étudions ensuite l'espace des HDVFs. L'ensemble des HDVFs peut être muni d'une structure de graphe dont les arêtes correspondent à des opérations élémentaires d'échange. À l'aide d'une preuve constructive, nous montrons que le graphe des HDVFs parfaits est connexe. De plus, nous proposons un algorithme permettant de parcourir ce graphe afin d'adapter un HDVF donné à une base explicite donnée.

Chapitre 4. Au-delà de l'Information Homologique Standard

Le Chapitre 4 (troisième contribution) explore des interactions plus profondes entre combinatoire et topologie algébrique. Nous utilisons le cadre combinatoire des HDVFs pour retrouver des résultats classiques de topologie algébrique. En particulier, nous proposons une version constructive et combinatoire de la dualité d'Alexander.

Cette approche motive l'introduction des *configurations d'homologie*, qui visent à classifier les HDVFs et fournissent une compréhension plus globale des calculs d'homologie. Nous démontrons qu'elles définissent un invariant combinatoire. En particulier, l'ensemble des configurations d'homologie de dimension maximale con-

stitue un invariant topologique pour les espaces de dimension au plus 3. Nous discutons également des aspects computationnels ainsi que des perspectives associées.

Nous concluons ce chapitre en reliant les deux travaux précédents. Nous montrons comment les configurations d'homologie aident à characteriser les isomorphismes d'homologie constructibles par HDVFs, en particulier ceux liés à la dualité d'Alexander.

Chapitre 5. Mesures Géométriques pour les Trous Homologiques

Le Chapitre 5 (quatrième contribution) déplace l'attention de l'homologie combinatoire vers l'interaction entre géométrie et topologie.

Contrairement à la topologie, qui étudie les propriétés à déformation continue près, la géométrie étudie les formes avec des notions de distance et de mesure, notamment à travers des questions de taille, de proximité, de courbure et de volume. Ainsi, topologie et géométrie étudient les mêmes objets, mais selon des points de vue différents.

L'intégration de géométrie dans l'homologie conduit à des questions telles que la localisation, la taille ou encore la position des trous. Ces questions se situent à l'intersection de la géométrie, de la topologie et de notre intuition humaine des trous. Elles motivent le travail de ce chapitre, où nous introduisons des mesures géométriques associées aux trous homologiques. Précisément, nous utilisons l'homologie persistante pour associer à chaque trou la plus grande boule qu'il contient, ainsi que la plus petite boule nécessaire pour l'ouvrir. Nous proposons des méthodes de calcul effectives ainsi que des résultats expérimentaux. Le chapitre se conclut par des perspectives autour de la définition et du calcul de la « forme » des trous homologiques.

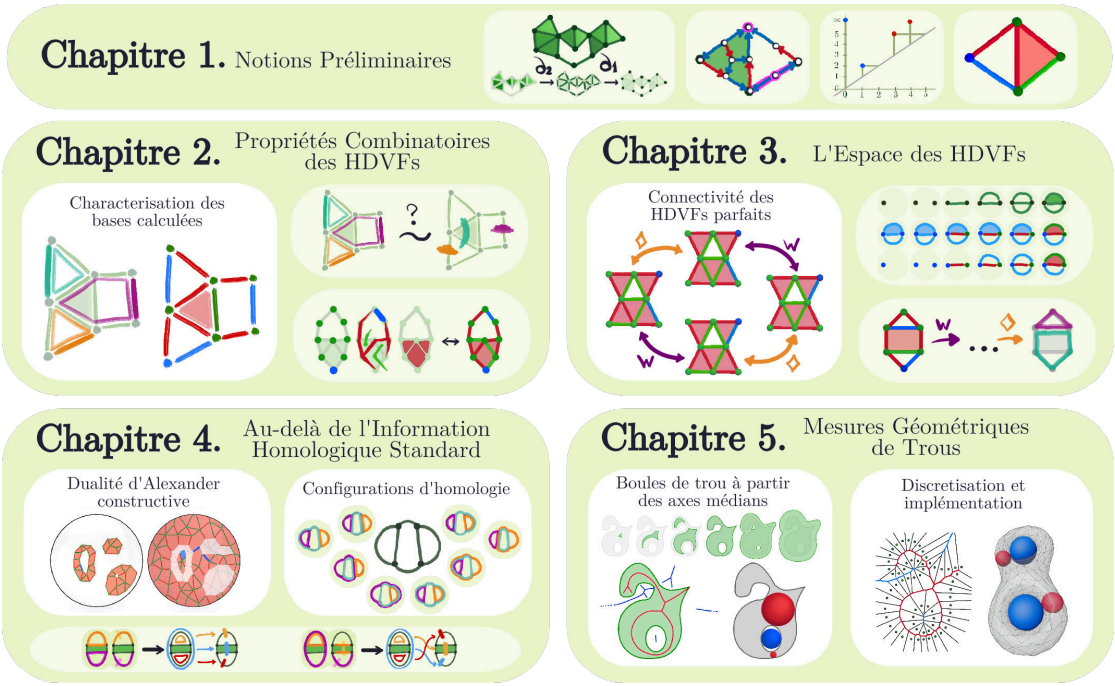


Figure 1 – Illustrations des chapitres et contributions de ce manuscrit.

Contribution Additionnelle. Stabilité et Extension de la Persistence Steady et Ranging

Cette dernière contribution n'est pas développée dans ce manuscrit, bien qu'elle ait été réalisée durant la période de doctorat. Elle a pour point de départ un séjour de recherche à l'Université de Bologna (Italie), au printemps 2023, effectué en collaboration avec Massimo FERRI *et Patrizio* FROSINI. *Un bref résumé est présenté ici, pour plus de détails, nous renvoyons à* GAZULL *2025*.

Au cours des dernières décennies, l'homologie persistante a été largement généralisée, étendant son champ d'application au-delà du domaine de la topologie. Parmi ses extensions, les persistances steady et ranging ont été développées afin d'étudier une grande variété de propriétés des graphes. Plus précisément, étant donnée une propriété d'intérêt sur les graphes, il est possible de construire deux types de persistance (la persistance steady et la persistance ranging) qui décrivent l'évolution de cette caractéristique le long de filtrations de graphes. Dans GAZULL 2025, nous étendons les persistances steady et ranging à d'autres objets à l'aide de la théorie des catégories et étudions la stabilité de telles persistances. En particulier, une caractérisation des propriétés qui induisent des persistances steady et ranging équilibrées est proposée. Les principaux résultats de ce travail sont illustrés à l'aide d'une implémentation dans le cas des hypergraphes.

Introduction (English)

Topology studies properties of shapes, considered as spaces, that are preserved under continuous deformations. These properties include connectivity, adjacency, loops, and holes. A central goal of topology is to determine whether two spaces are equivalent up to continuous deformation. Two such spaces are said to be *homeomorphic*.

To address this question, mathematicians have introduced *topological invariants*. They are mathematical objects, defined on spaces, that remain equivalent on homeomorphic spaces. The study of such algebraic invariants is the main subject of algebraic topology. Among them, *homology*, the main character of this manuscript, intuitively captures information about the holes of a space. A strong intuition is that spaces with different types or numbers of holes are not homeomorphic.

The interaction between combinatorics and topology arises when a space is decomposed into elementary pieces. Such a decomposition is called a discretization. Studying topology through discretizations has led to many fruitful ideas. A classical example is the Euler characteristic, which captures topological information by simply counting the cells of a discretization.

From a historical point of view, algebraic topology was primarily named *combinatorial topology* during the first half of the 20th century. Many results were conceived in the setting of triangulated piecewise-linear manifolds. This name reflected the approach of understanding spaces through their discretizations. Later, in the mid-century, the field shifted towards algebra, following the development of new algebraic tools, such as category theory. More recently, interactions between combinatorics and topology have led to the field of computational topology, in particular computational homology, which now finds many applications in topological data analysis.

This thesis explores homology and its computations, mainly from a combinatorial point of view, but also from a geometric one. More generally, it investigates several interactions between topology, combinatorics, and geometry.

A significant part of this thesis focuses on the interaction between combinatorics and homology, under the lens of *Homological Discrete Vector Fields* (HDVFs). HDVFs are hybrid objects, in between combinatorics and algebra, that encode homological information. They emerged as a generalization of discrete Morse theory, with ideas from effective homology. Throughout this thesis, HDVFs are both considered as objects of study and as theoretical tools for investigating homology computations. We now briefly present the chapters. See Figure 1 for an illustration of each of them.

Chapter 1. Common Background

Chapter 1 introduces the background material and the scientific context of this

manuscript. We begin with an introductory example. We then present standard discretizations of topological spaces, and introduce the theory of homology. In particular, we present chain complex constructions derived from spaces and their discretizations. Next, we review computational homology approaches, as well as persistent homology. We conclude this chapter with an introduction to HDVFs.

Chapter 2. Combinatorial Properties of HDVFs

Chapter 2 (first contribution) studies the properties of HDVFs from a "close" point of view. First, we present *tri-partitions*, introduced by H. EDELSBRUNNER and ÖLSBÖCK 2020, which are combinatorial objects that capture homological information. We prove that tri-partitions and perfect HDVFs represent the same concept, but respectively adopt a horizontal and a vertical point of view. This equivalence provides new insights and improves the understanding of HDVFs as a combinatorial structure.

Second, we characterize the homology and cohomology bases computed by HDVFs. Precisely, we introduce *explicit* (co)homology bases, which generalize the strictly fundamental bases from graph theory. These bases combine both combinatorics and algebra, and we show that they are precisely the homology bases computed by HDVFs. This characterization naturally transfers to related computational homology methods.

Finally, we establish a necessary and sufficient condition for a pair of explicit homology and cohomology bases to be induced by a single HDVF.

Chapter 3. The Space of HDVFs and Homology Computations

Chapter 3 (second contribution) adopts a more "global" perspective on HDVFs and homology computations. We begin by developing a deeper understanding of the relationship between persistent homology and HDVFs. We show that every filtration induces a unique HDVF encoding its persistent homology information. Conversely, every HDVF is induced by at least one filtration.

We then study the space of HDVFs itself. The set of HDVFs can be equipped with a graph structure, whose edges correspond to elementary swapping operations. Using a constructive proof, we show that the graph of perfect HDVFs is connected. In addition, we provide an algorithm that navigates this graph to adapt a given HDVF to a given explicit basis.

Chapter 4. Beyond Standard Homology Information

Chapter 4 (third contribution) explores deeper interactions between combinatorics and algebraic topology. We first recover classical algebraic-topological results with the combinatorial framework of HDVFs. In particular, we derive a constructive and combinatorial version of Alexander duality.

This approach motivates the introduction of *homology configurations*. Homology configurations aim to classify homology computations, to provide a broader understanding of the possible computations given a discretized space. We prove that they define a combinatorial invariant. In particular, this implies that the set of homology configurations of maximal dimension is a topological invariant for spaces of

dimension at most 3. We also discuss computational aspects and related prospects.

We conclude this chapter by connecting the two previous works. We show how homology configurations help to describe the homology isomorphisms that can be constructed using HDVFs, in particular the Alexander duality isomorphisms.

Chapter 5. Geometric Measures for Homological Holes

Chapter 5 (fourth contribution) shifts the focus from combinatorial homology to the interaction between geometry and topology.

Unlike topology, which studies properties up to continuous deformations, geometry studies shapes through notions of distance and measurement. It considers questions related to size, proximity, curvature and volume. Hence, topology and geometry often study the same objects, but they do so with different points of view.

Incorporating geometric information into homology naturally leads to questions such as the location, size and relative position of holes. These questions confront geometry, topology, and our human intuition about holes. This motivates the work presented in this chapter, in which we introduce geometric measures associated with homological holes. Precisely, for each homological hole, we use persistent homology to define the largest ball contained in it, and the smallest ball necessary to open it. We propose practical methods for computing these hole balls, and experimental results.

We conclude this chapter with ongoing works about defining and computing the "shape" of homological holes.

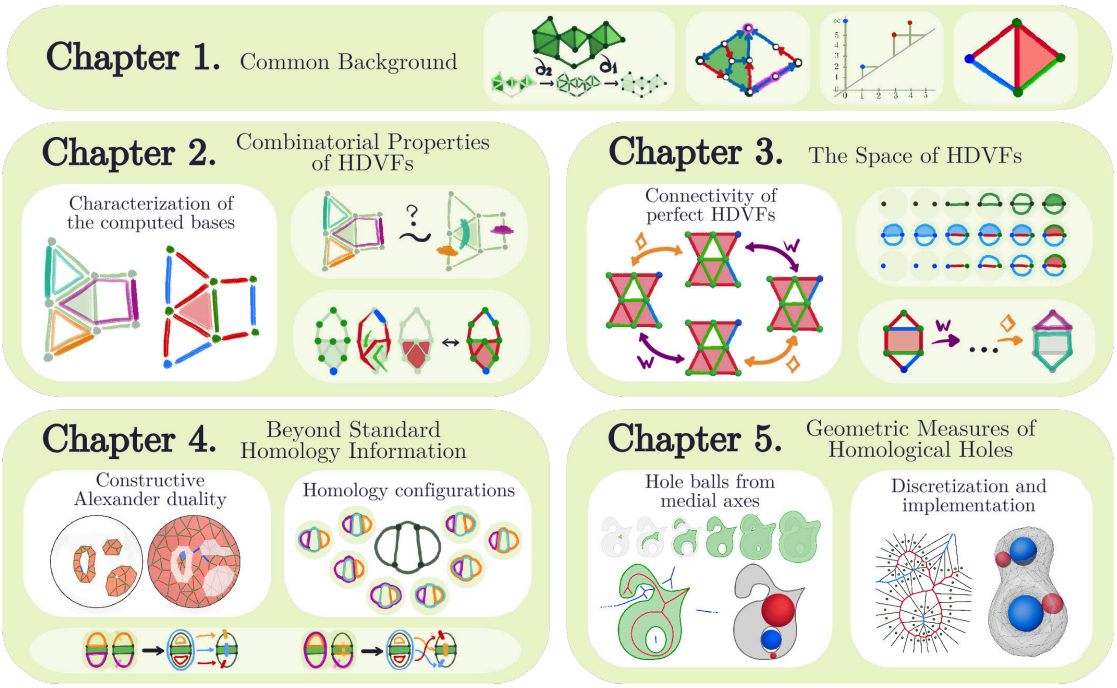


Figure 1 – Illustrations of the chapters and contributions of this manuscript.

Additional Contribution. Stability and Extension of Steady and Ranging Persistence

This last contribution is not developed in this manuscript even though it was conducted during the PhD period. It started during a research stay in the university of Bologna (Italy), in spring 2023, working with Massimo FERRI *and Patrizio* FROSINI. *Here we present a short summary, please refer to* GAZULL *2025 for more details.*

During the last decades, persistent homology has been widely generalized, extending its scope beyond the field of topology. Among its extensions, steady and ranging persistences were developed to study a wide variety of graph properties. Precisely, given a feature of interest on graphs, it is possible to build two types of persistence (steady and ranging persistence) that follow the evolution of the feature along graph filtrations. In GAZULL 2025, we extend steady and ranging persistence to other objects using category theory and investigates the stability of such persistence. In particular, a characterization of the features that induce balanced steady and ranging persistence is provided. The main results of this study are illustrated using a practical implementation of the theory for hypergraphs.

1. Common Background

Sommaire

1.1. An Introductory Example . . . 22
1.2. Geometric Complexes: Discretizations of Topological Spaces . . . 25
1.2.1. Simplicial complexes . . . 25
1.2.2. CW-complexes . . . 26
1.2.3. Other geometric complex constructions . . . 27
1.3. Chain Complexes and Homology . . . 27
1.3.1. General chain complexes . . . 28
1.3.2. Homology groups . . . 29
1.3.3. Singular chain complexes . . . 31
1.3.4. Introducing combinatorics: combinatorial chain complexes . . . 32
1.3.5. Geometric chain complexes . . . 34
1.3.5.1. Simplicial chain complexes . . . 35
1.3.5.2. Cellular chain complexes . . . 36
1.3.5.3. Other geometric chain complexes . . . 37
1.3.6. Intuition, holes and homology bases . . . 37
1.3.7. Cohomology . . . 38
1.3.8. A word about homotopy . . . 40
1.4. Computing Homology . . . 40
1.4.1. The Smith normal form . . . 42
1.4.2. Reductions and effective homology . . . 43
1.4.3. Discrete Morse theory and DGVFs . . . 44
1.5. Persistent Homology . . . 45
1.5.1. Definition . . . 46
1.5.2. Stability . . . 47
1.5.3. Computing persistent homology . . . 47
1.5.4. Extensions of persistent homology . . . 48
1.6. Homological Discrete Vector Fields . . . 49
1.6.1. Definition and main properties . . . 50
1.6.2. Elementary swapping operations . . . 56
1.6.3. Completion of HDVF . . . 60
1.6.4. Emulating standard computational homology methods . . . 62

1.1. An Introductory Example

In this work, we study shapes, which are formalized in mathematics as *topological spaces*. Shapes can be studied from different points of view. For instance, geometry studies shapes through notions such as distance, curvature, area and volume. By contrast, topology studies properties that are preserved under continuous deformation of the shape, such as connectivity, loops and holes. In this thesis, we mainly study holes, which constitutes a topological property. Precisely, we investigate holes from algebraic, combinatorial and geometric points of view.

A fundamental concept for studying holes in the "algebraic world" is *homology*. Within the framework of homology, holes have an associated dimension. Depending on the dimension, the holes correspond to different intuitive features. For instance, 0, 1, and 2-dimensional holes correspond to connected components, tunnels (or handles), and cavities (voids), respectively (see Figure 1.2).

To obtain homological information from a topological space, we first translate the space into an algebraic structure called *chain complex*. A chain complex consists of a sequence of vector spaces (or modules) K_q together with boundary operators $\partial_q : K_q \to K_{q-1}$, that satisfy the relation $\partial_q \circ \partial_{q+1} = 0$. Every chain complex induces a collection of algebraic objects called *homology groups*, which encode information about the holes of the underlying space.

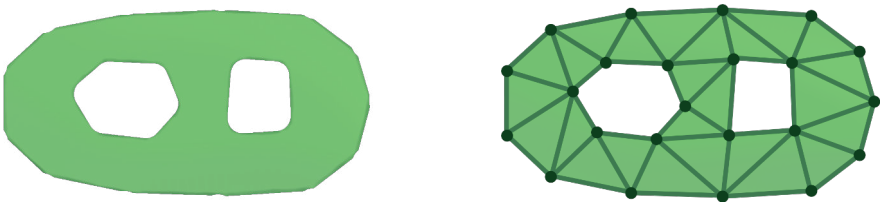


Figure 1.1 – A shape (left) and a discretization of it (right).

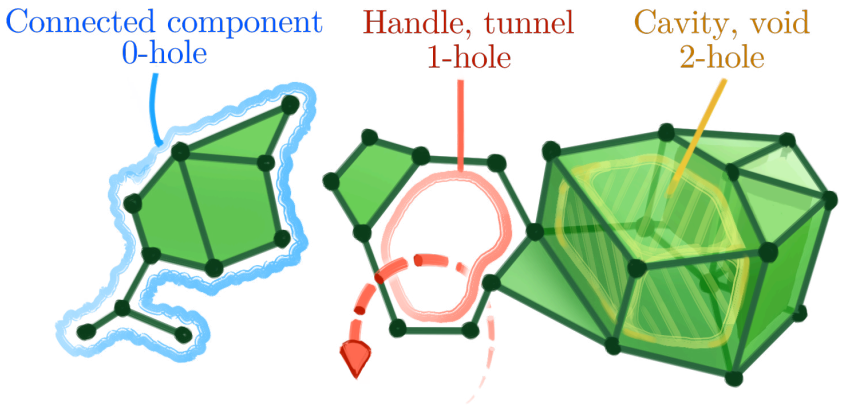


Figure 1.2 – Illustration of the intuition behind homology.

There exist several ways of building a chain complex from a topological space. The

most general construction is the *singular chain complex*. However, singular chain complexes are usually infinite and therefore not suitable for computation.

A computable alternative is to construct a chain complex from a *discretization* of the space. A discretization is a finite representation of a topological space, composed of elementary building blocks, called *cells*. For example, Figure 1.1 shows a continuous shape X together with a discretization of X using vertices, edges, and triangles. Such a discretization is usually called a *mesh* in geometry, a *simplicial complex* in algebraic topology, or more generally a *triangulation*.

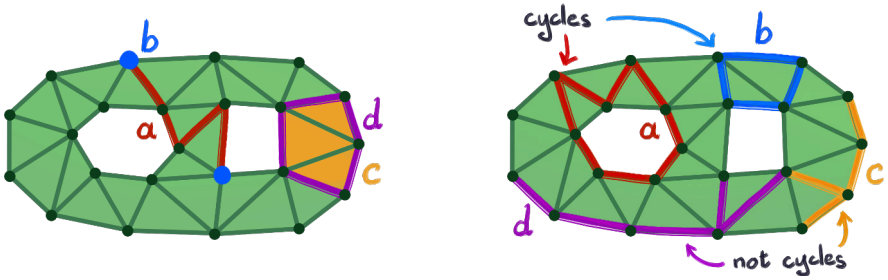


Figure 1.3 – An illustration of chains, cycles, and boundaries. In the left image, a is the sum of the red edges. The chain b, the sum of the two blue vertices, is the boundary of a, that is, $\partial(a) = b$. Similarly, $d = \partial(c)$. The right image shows chains that are cycles (a and b) and chains that are not cycles (c and d).

In this setting, a *chain* in K_q is a formal sum of q-dimensional cells of the discretization, which intuitively encode paths, see Figure 1.3 for an example. The boundary operator ∂ is defined to capture the notion of the boundary of a face. For instance, the boundary of a triangle is the sum of its three edges.

A *cycle* is a chain x whose boundary $\partial(x)$ is equal to zero. Intuitively, 1-dimensional cycles correspond to loops, whereas 2-dimensional cycles correspond to closed surfaces. Now, a *boundary* is a chain that can be written as the boundary of another chain, see for instance the chain b in Figure 1.3.

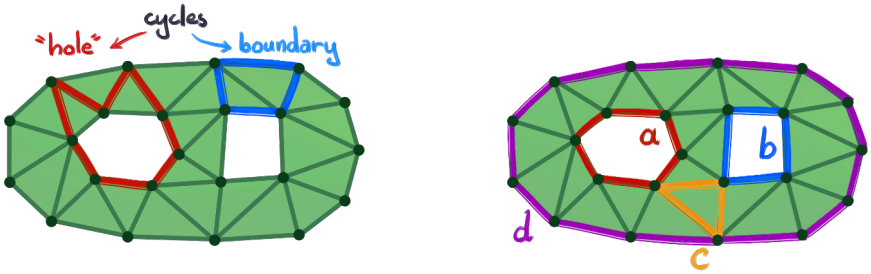


Figure 1.4 – An illustration of holes and homology groups. The left image shows a cycle that is a boundary, and a cycle that surrounds a hole. The right image represents the homology group of dimension 1. Each homology class is represented with a cycle: The cycles a and b surround the left and right holes, respectively. The cycle d surrounds both holes. The cycle c does not surround any hole, as it is a boundary.

Since the boundary operators satisfy $\partial_q \circ \partial_{q+1} = 0$, every boundary is necessarily a cycle. Alternatively, the boundary of a boundary is zero. Cycles that are not boundaries intuitively "surround" a hole (or several holes). Homology groups are obtained by identifying cycles that differ by a boundary. Thus, homology groups intuitively classify the different ways of "surrounding" holes, see Figure 1.4.

The remainder of this chapter develops these ideas more formally. Next section introduces common discretizations of topological spaces, including simplicial complexes and CW-complexes, which are gathered under the term *geometric complex*. The following section presents the theory of homology. In particular, it explains how topological spaces are translated into chain complexes, both from continuous and discrete points of view.

The concepts of both these sections are illustrated in Figure 1.5. In particular, it distinguishes the realm of topology from the realm of algebra, and displays the common relations between them.

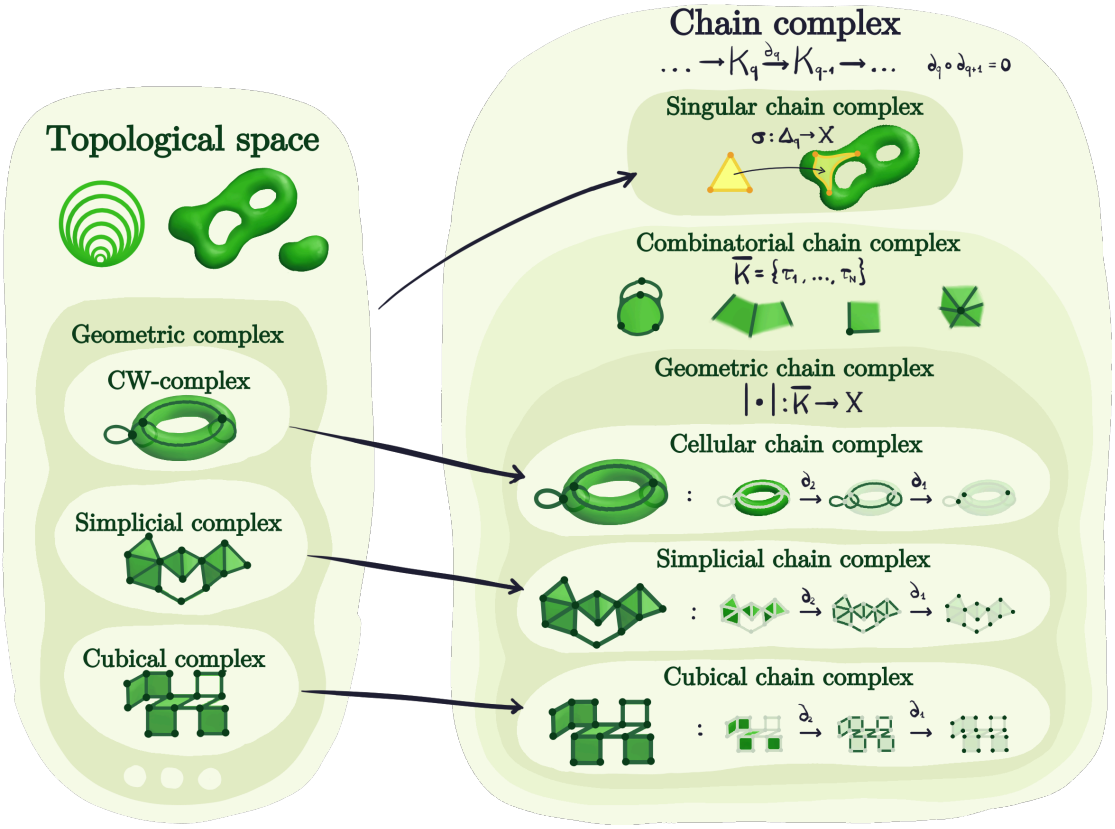


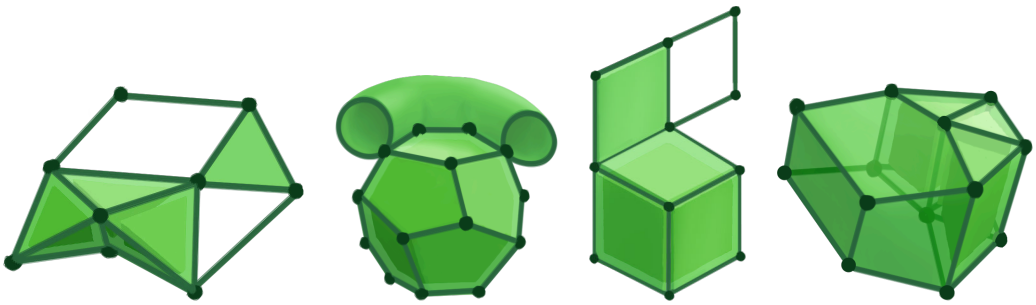
Figure 1.5 – The topological and algebraic concepts introduced in the following sections. Section 1.2 introduces concepts on the left part, precisely *geometric complexes*, which correspond to discretizations of topological spaces. Section 1.3 introduces the right part, in particular how geometric complexes translate into their algebraic counterparts, *geometric chain complexes*.

The chain complexes that are considered in the remaining chapters may vary: Chap-

ter 2 and Chapter 3 are exclusively about combinatorial chain complexes. Chapter 4 mainly considers combinatorial chain complexes, but some results only apply to geometric chain complexes or simplicial chain complexes. Chapter 5 considers singular chain complexes and geometric chain complexes, as it is about both continuous and discretized geometric objects.

1.2. Geometric Complexes: Discretizations of Topological Spaces

In this part, we present several discrete constructions of topological spaces. We begin with *simplicial complexes*, whose building blocks are simplices (points, edges, triangles, tetrahedra, etc.). Then, we introduce the more general and more abstract notion of a *CW-complex*. We conclude with a brief discussion about related constructions, including cubical and polyhedral complexes. We gather all these constructions under the term *geometric complex*, see Figure 1.6.



(a) (b) (c) (d)

Figure 1.6 – A simplicial complex (a), a CW-complex (b), a cubical complex (c) and a polyhedral complex (d). They all can be considered as geometric complexes. Note that simplicial, cubical and polyhedral complexes prohibit multi-incidence, whereas CW-complexes allow it.

1.2.1. Simplicial complexes

Let $v_0, \dots, v_q$ be points in $\mathbb{R}^N$, for some fixed N, and denote by $[v_0, \dots, v_q]$ their convex hull. If these points are in general position (i.e., the family $\{v_i - v_0\}_{i=1}^{q}$ is linearly independent), $[v_0, \dots, v_q]$ is called a *q-simplex*. Intuitively, 0-simplices are points, 1-simplices are edges, 2-simplices are triangles, and 3-simplices are tetrahedra.

The *faces* of a simplex $[v_0, \dots, v_q]$ are the simplices obtained from all subsets of $\{v_0, \dots, v_q\}$. We denote by $\Delta[v_0, \dots, v_q]$ the set of all faces of $[v_0, \dots, v_q]$. For each $i \in 0, \dots, q$, we denote by $[v_0, \dots, \hat{v}_i, \dots, v_q]$ the simplex obtained by removing the vertex v_i. Such a simplex is called a *facet* of $[v_0, \dots, v_q]$. The set of facets of a q-simplex

$[v_0, \dots, v_q]$ is the set:

$$\delta[v_0, \dots, v_q] = \left\{ [v_0, \dots, \hat{v}_i, \dots, v_q] \;\middle|\; i = 0, \dots, q \right\}$$

Definition 1.2.1 (simplicial complex)**.** A *simplicial complex* is a finite set of simplices X satisfying the following conditions:

1. Every face of a simplex in X also belongs to X.
2. The intersection of any two simplices in X is either empty or a common face of both simplices. Precisely, if $\tau, \tau' \in X$ satisfy $\tau \cap \tau' \neq \emptyset$, then $\tau \cap \tau'$ is a simplex that is a face of both τ and τ'.

X is naturally associated to the topological space given by the union of all its simplices. See Figure 1.6 for an illustration of a simplicial complex.

A *simplicial subcomplex* of X is a subset $X' \subseteq X$ that is still a simplicial complex.

1.2.2. CW-complexes

We now introduce CW-complex[1]. Contrary to simplicial complexes, the cells of CW-complexes are not required to be simplices, but can be arbitrary topological spaces that are homeomorphic to disks. A CW-complex is defined inductively with respect to the dimension. Precisely, we construct a sequence of topological spaces $(X^q)_{q \geq 0}$, called the *q-skeletons*, where each X^q has dimension q. See Figure 1.7 for an illustration of the process.

1. We start with a finite set of points $\mathscr{C}_0 = X^0$, whose elements are called 0-cells.
2. Given X^q, we construct X^{q+1} by attaching $(q+1)$-cells to X^q. Let $\mathscr{C}_{q+1}$ be a finite set whose elements are called $(q+1)$-cells. For each $\tau \in \mathscr{C}_{q+1}$, we associate a set B_τ homeomorphic to the open $(q+1)$-dimensional ball $\mathbb{B}^{q+1}$.

   The boundary of B_τ is defined as $\partial B_\tau = \overline{B_\tau} \setminus B_\tau$, where $\overline{B_\tau}$ denotes the closure of B_τ, and is homeomorphic to the q-sphere $\mathbb{S}^q$. Each cell $\tau \in \mathscr{C}_{q+1}$ is attached to X^q via a continuous map $\phi_\tau : \partial B_\tau \to X^q$, called the *gluing map*.

   The space X^{q+1} is then defined as the quotient of the disjoint union of X^q and $\bigsqcup_{\tau \in \mathscr{C}_{q+1}} \overline{B_\tau}$ under the identification $x \sim \phi_\tau(x)$ for all $x \in \partial B_\tau$ and all $\tau \in \mathscr{C}_{q+1}$.
3. This process is iterated up to a fixed dimension n, yielding the space $X = X^n$.

As a set, X is the disjoint union of the open cells B_τ for all $\tau \in \mathscr{C}_q$ and all q (where B_τ is simply a point when $\tau \in \mathscr{C}_0$).

CW-complexes form a large class of topological spaces that generalize many discrete constructions. In particular, polyhedra and differentiable manifolds admit a CW-complex structure.

A *CW-subcomplex* of X is a closed subset $X' \subseteq X$ that is a union of open cells B_τ, for τ in a subset of $\bigcup_q \mathscr{C}_q$. The fact that X' is closed ensures that whenever a cell belongs

[1] For simplicity, we restrict our definitions to finite-dimensional CW-complexes with finitely many cells.

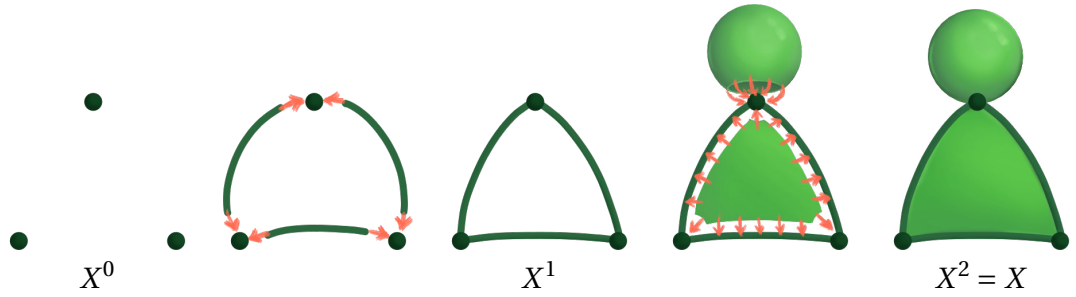


Figure 1.7 – Illustration of a CW-complex construction process.

to X', all cells to which it is attached also belong to X'. Thus, a CW-subcomplex X' inherits the CW-complex structure from X.

1.2.3. Other geometric complex constructions

Other geometric complexes have been introduced in the literature.

A common example is *cubical complexes*, whose cells are cubes of various dimensions in a regular grid. In digital topology, cubical cells are often encoded using Khalimsky coordinates, which provide a useful representation. Another example is provided by Δ*-complexes*, which generalize simplicial complexes by allowing more flexible "gluing" options. A more general construction is given by *polyhedral complexes*, which allow arbitrary polyhedra. Polyhedral complexes include for instance *linear cell complexes* (DAMIAND 2026), whose combinatorics is encoded using a combinatorial map or a generalized map.

In the next section, we present how topological spaces can be translated into algebraic objects called *chain complexes*. In the specific case of geometric complexes, these chain complexes are referred to as *geometric chain complex*. These algebraic structures encode topological information and allows us to define and compute homology, which intuitively captures the holes of the topological space.

1.3. Chain Complexes and Homology

We now shift our focus from topological spaces to the algebraic structures used to study them.

We start with the general notion of *chain complex* and homology groups. We then introduce *singular chain complexes*, which provide the most general way of translating a topological space into a chain complex. Although singular chain complexes are useful theoretically, they are usually infinite and therefore not suitable for computations. Hence, we introduce *combinatorial chain complexes*, which form a computationally suited class of chain complexes, that also incorporate combinatorics via the notion of cell. Finally, we introduce *geometric chain complexes*, which are combinatorial chain complexes built from the geometric complexes introduced in Section 1.2. In particular

this class includes *simplicial chain complexes* and *cellular chain complexes*.

After introducing all these chain complexes constructions, we discuss the intuition underlying homology, and briefly introduce the related notions of cohomology and homotopy.

Homology is defined with coefficients in a fixed ring $\mathscr{R}$. The information captures by homology depends on the choice of coefficients. On one hand, when $\mathscr{R}$ is a field, homology intuitively provides the number holes. On the other hand, when $\mathscr{R}$ is not a field, homology may contain additional information, known as *torsion*. In the absence of torsion, we obtain the same information, regardless of the specific choice of ring. In practice, homology is computed with coefficients in either the ring $\mathbb{Z}$ or the field $\mathbb{Z}/2\mathbb{Z}$.

In this section, we first present the general definition with coefficients in an arbitrary ring $\mathscr{R}$. Later, we will restrict to homology with coefficients in a field, in particular with $\mathscr{R} = \mathbb{Z}/p\mathbb{Z}$. For more details about homology and algebraic topology, please refer to HATCHER 2002.

Notations. We fix a ring $\mathscr{R}$. Given a $\mathscr{R}$-module A and a submodule $B \subseteq A$, we denote by $\mathtt{i}_B$ the inclusion map from B to A. Similarly, we denote by $\mathtt{j}_B$ the projection map from A to B. The identity map on A is denoted by id_A or simply id.

The compositions of two compatible maps ϕ and ψ is denoted $\phi \circ \psi$, or simply $\phi\psi$. Moreover, when no confusion arises, we sometimes write $\phi\tau$ instead of $\phi(\tau)$, for an element τ.

Depending on the context, the symbol "0" can mean the zero element of a $\mathscr{R}$-module, a zero map (that sends every element to zero), or simply the trivial group $\{0\}$.

1.3.1. General chain complexes

The definition of homology is purely algebraic and is formulated in terms of algebraic objects called chain complexes.

Definition 1.3.1 (chain complex)**.** A *chain complex* (K, ∂) is a sequence of $\mathscr{R}$-modules $(K_q)q \in \mathbb{Z}$ together with maps $\partial_q : K_q \to K_{q-1}$ such that $\partial_q \circ \partial_{q+1} = 0$ for all q.

The module K_q is called the *q-chain group*, and its elements are called *q-chains*. The map ∂_q is called the *q-boundary map*. The elements of $\ker \partial_q$ are called the *q-cycles*, whereas the elements of $\operatorname{im} \partial_{q+1}$ are called the *q-boundaries*.

In many situations, $K_q = \{0\}$ for $q < 0$ and $q > n$, for some integer n. In this case, a chain complex is represented as

$$\xrightarrow{0} K_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} K_q \xrightarrow{\partial_q} K_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} K_0 \xrightarrow{0}$$

Often, we omit the dimension and consider the direct sum $K = \bigoplus_{q=0}^{n} K_q$, whose elements are simply called *chains*. Similarly, the map $\partial = \bigoplus_{q=0}^{n} \partial_q : K \to K$ is called the *boundary operator*.

We often consider chain complexes that are obtained as a restriction of another larger chain complexes. In this case, we call them chain *subcomplexes*.

Definition 1.3.2 (chain subcomplex)**.** Let (K,∂) be a chain complex. (K',∂') is a *chain subcomplex* of (K,∂) if (K',∂') is a chain complex, K'_q is a submodule of K_q for all q, and ∂'_q is the restriction of ∂_q to K'_q, i.e. $\partial'_q = \mathtt{j}_{K'_{q-1}} \circ \partial_q \circ \mathtt{i}_{K'_q}$.

We now introduce *chain morphisms*, which are morphisms between such chain complexes.

Definition 1.3.3 (chain morphism)**.** A *chain morphism* (or *chain map*) $\phi : (K,\partial) \to (K',\partial')$ is a sequence of maps $\phi_q : K_q \to K'_q$ for all q such that $\partial'_q \circ \phi_q = \phi_{q-1} \circ \partial_q$.

Equivalently, the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & K_{q+1} & \xrightarrow{\partial_{q+1}} & K_q & \xrightarrow{\partial_q} & K_{q-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \phi_{q+1}} & & \downarrow{\scriptstyle \phi_q} & & \downarrow{\scriptstyle \phi_{q-1}} & & \\
\dots & \longrightarrow & K'_{q+1} & \xrightarrow{\partial'_{q+1}} & K'_q & \xrightarrow{\partial'_q} & K'_{q-1} & \longrightarrow & \dots
\end{array}$$

In particular, there is a canonical chain morphism from a subcomplex to its parent complex, defined by the inclusion maps.

1.3.2. Homology groups

The condition $\partial_q \circ \partial_{q+1} = 0$ implies that $\operatorname{im}\partial_{q+1} \subseteq \ker\partial_q$, that is, every q-boundary is a q-cycle. We can therefore define an equivalence relation on q-cycles by declaring two cycles equivalent if their difference is a boundary:

$$\text{for } x, x' \in \ker\partial_q, \qquad x \sim x' \iff x - x' \in \operatorname{im}\partial_{q+1}$$

The equivalence classes of this relation define the homology groups.

Definition 1.3.4 (homology group)**.** The *q-homology group* of a chain complex (K,∂), denoted $H_q(K;\mathscr{R})$, is defined as

$$H_q(K;\mathscr{R}) = \ker\partial_q / \sim \qquad \text{equivalently:} \qquad H_q(K;\mathscr{R}) = \ker\partial_q / \operatorname{im}\partial_{q+1}$$

When they are finitely generated, homology groups have a known form. Precisely, when $\mathscr{R}$ is a field, the q-th homology group is a vector space of the form $\mathscr{R}^{\beta_q(K;\mathscr{R})}$, where $\beta_q(K;\mathscr{R})$ is called the *q-th Betti number* of K. When $\mathscr{R}$ is a general ring, $H_q(K;\mathscr{R})$ is more informative:

Proposition 1.3.1. *Homology groups can be decomposed as follows:*

$$H_q(K;\mathscr{R}) \;\approx\; \mathscr{R}^{\beta_q(K;\mathscr{R})} \;\oplus\; \mathscr{R}/\alpha_1\mathscr{R} \;\oplus \cdots \oplus\; \mathscr{R}/\alpha_r\mathscr{R}$$

$\mathscr{R}^{\beta_q(K;\mathscr{R})}$ is called the free *part, whereas $T_q = \mathscr{R}/\alpha_1\mathscr{R} \oplus \cdots \oplus \mathscr{R}/\alpha_r\mathscr{R}$, a sum of cyclic groups, is called the* torsion *part of $H_q(K;\mathscr{R})$.*

A fundamental result is that the homology groups computed with $\mathscr{R} = \mathbb{Z}$ completely determine the homology groups computed with other coefficients. This is known as the *universal coefficient theorem*, see HATCHER 2002.

> *Remark.* In the presence of torsion, $\beta_q(K;\mathscr{R})$ may depend on the choice of $\mathscr{R}$. A simple example is the Klein bottle K, where we have $\beta_2(K;\mathbb{Z}_2) = 1$, whereas $\beta_2(K;\mathbb{Z}) = \beta_2(K;\mathbb{Z}_3) = 0$. Though, as stated in Corollary 3A.6 of HATCHER 2002, it is always possible to recover $H_q(K;\mathscr{R})$ from $H_q(K;\mathbb{Z})$.

Notations. In the whole manuscript, we omit mentioning $\mathscr{R}$ and simply write $H_q(K)$ and $\beta_q(K)$. Sometimes we also write β_q instead of $\beta_q(K)$ when no confusion arises. In most of our works, we will set $\mathscr{R}$ to be a field (for instance $\mathbb{Z}/p\mathbb{Z}$).

We denote by $[\cdot]_q^K : \ker\partial_q \to H_q(K)$ the canonical projection sending a q-cycle to its homology class, and by $s_q^K : H_q(K) \to \ker\partial_q$ a map that assigns to each homology class a representative cycle. In particular, $[\cdot]_q^K \circ s_q^K = id$, and for all $x \in \ker\partial_q$ there exists $y \in K_{q+1}$ such that $s_q^K[\cdot]_q^K(x) = x + \partial_{q+1}(y)$. For clarity, we sometimes omit the superscript K, or the subscript q, when there is no confusion.

A chain morphism $\phi : (K,\partial) \to (K',\partial')$ induces homomorphisms on homology groups. More precisely, for each q we define $H_q\phi : H_q(K) \to H_q(K')$ by $H_q\phi = [\cdot]_q^{K'} \circ \phi_q \circ s_q^K$. If each ϕ_q is an isomorphism, then K and K' have isomorphic homology groups.

> *Remark.* From a categorical point of view, it is possible to consider the category whose objects are chain complexes and whose morphisms are chain morphisms. Homology then defines a functor from this category to the category of groups, sending chain complexes to their homology groups and chain morphisms to the induced group homomorphisms.

At first glance, the definition of chain complexes and homology is purely algebraic and its connection with topology is not immediate. This connection arises when we encode the topological structure of a space into a chain complex. The homology of the resulting chain complex then encodes topological information about the space.

Several such constructions have been developed, and are presented in this chapter. We begin with singular homology, which provides the most general construction of homology for topological spaces. Then, Section 1.3.4 introduces a specific class of chain complexes, equipped with a notion of *cell*, called *combinatorial chain complexes*. This framework generalizes chain complexes derived from geometric complexes, which we refer to as *geometric chain complexes*, and which are presented in Section 1.3.5. In particular, *geometric chain complexes* includes *simplicial chain complexes*, which are derived from simplicial complexes, and *cellular chain complexes*, which are derived from CW-complexes.

Importantly, the focus of this manuscript is on the homology of finite geometric and combinatorial representations of topological spaces.

Remark. Other approaches exist, including Čech and de Rham cohomology, but we do not cover them in this manuscript. More generally, since chain complexes and homology are purely algebraic, they can also be defined in contexts unrelated to topology, such as group homology.

1.3.3. Singular chain complexes

We describe here a general construction that associates a chain complex to a topological space X. This construction is based on the notion of simplex, defined in Section 1.2.1.

Let Δ^q be the standard q-simplex, defined by $\Delta^q = [e_0, \dots, e_q]$, where $e_0, \dots, e_q$ are the vectors of the canonical basis of $\mathbb{R}^{q+1}$.

Let X be a topological space. A *singular q-simplex* in X is a continuous map $\sigma : \Delta^q \to X$, see Figure 1.8 for an illustration. The boundary of σ, denoted $\partial_q(\sigma)$, is defined as a formal sum of singular $(q-1)$-simplices obtained by restricting σ to the facets of Δ^q. To ensure that $\partial_q \circ \partial_{q+1} = 0$ and take into account orientation, we need to add signs. More precisely,

$$\partial_q(\sigma) = \sum_{i=0}^{q} (-1)^i \sigma \circ \mathtt{i}_{[e_0,\dots,\hat{e}_i,\dots,e_q]}.$$

Definition 1.3.5 (singular chain complex)**.** Given a topological space X, let K_q be the free $\mathscr{R}$-module generated by the singular q-simplices in X. The boundary operator ∂_q extends linearly to a map from K_q to K_{q-1}. (K_q, ∂_q) thus forms a chain complex, called the *singular chain complex* of X.

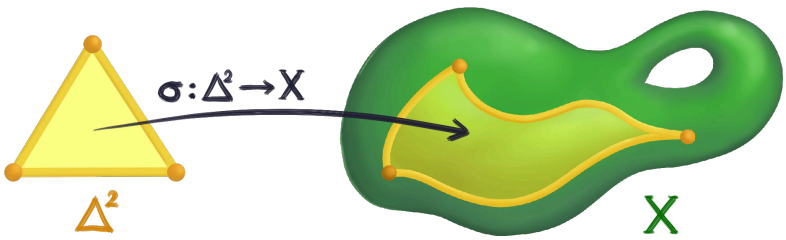


Figure 1.8 – Illustration of a singular 2-simplex $\sigma : \Delta^2 \to X$.

The q-th homology group of this chain complex is called the *singular homology group* of X and is denoted by $H_q(X)$.

From a computational perspective, singular homology is not directly computable, as it involves chain modules of infinite dimension. The next section introduces *combinatorial chain complexes*, which are finite-dimensional and therefore more suitable for computation.

1.3.4. Introducing combinatorics: combinatorial chain complexes

A *combinatorial chain complex* is a chain complex equipped with a specific choice of generating elements, called *cells*. Formally, it is a chain complex whose modules of q-chains K_q are generated by a finite and fixed set $\overline{K_q}$ of linearly independent elements, called q-cells.

Definition 1.3.6 (combinatorial chain complex)**.** A *combinatorial chain complex* (K,∂) is a chain complex together with, for each q, a finite set $\overline{K_q}$ forming a basis of K_q. We write $K_q = \operatorname{Span}\overline{K_q}$. The elements of $\overline{K_q}$ are called the *q-cells*. We denote by $\overline{K} := \bigsqcup_{q=0}^{n} \overline{K_q}$ the set of all cells.

> *Remark.* Combinatorial chain complexes are also referred to as *free chain complexes with a basis*, see for instance KOZLOV 2005. In SELLERS 2013, which is about theory of algorithms, they are simply referred to as *combinatorial complexes*. Though, this later term is currently used for another structure, see for instance HAJIJ et al. 2023, that is why we choose the term *combinatorial chain complexes* for the rest of this manuscript.

The choice of cells allows us to incorporate combinatorial structures to the chain complex. In particular, it induces a canonical bilinear form $\langle\cdot,\cdot\rangle : K \times K \to \mathscr{R}$, defined on the cells by $\langle\tau,\tau\rangle = 1$ and $\langle\tau,\tau'\rangle = 0$ for $\tau \neq \tau'$ in $\overline{K}$, and extended by bilinearity.

This allows the definition of the *associated cells* of a q-chain x as the set

$$\overline{x} := \{\tau \in \overline{K_q} \mid \langle x,\tau\rangle \neq 0\}$$

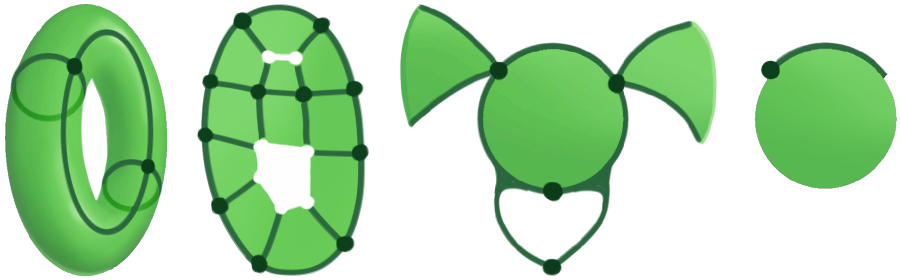
If $\tau \in \overline{x}$, we say that τ is a cell of x. If $\tau \in \overline{\partial(x)}$, we say that τ is a *facet* of x.

Given a chain complex, different choices of bases may lead to different combinatorial chain complexes, with potentially different combinatorial properties. However, in most practical situations, there is a canonical choice of cells induced by the construction of the chain complex. This is the case for standard examples such as simplicial chain complexes and cellular chain complexes, discussed in the next subsection.

In these common cases, the combinatorics of cells admits an intuitive interpretation. For example, 1-dimensional combinatorial chain complexes correspond to graphs, where 1-cells represent edges and their facets correspond to their endpoint 0-cells. However, more general combinatorial chain complexes may have less intuitive behavior. For instance, a 1-cell may have only one, or strictly more than two facets. This phenomenon is an important subtle point and is illustrated in Figure 1.9.

We now introduce the notion of *combinatorial chain subcomplex*, which is a chain subcomplex that preserves the notion of cell.

Definition 1.3.7 (combinatorial chain subcomplex)**.** Let (K,∂) be a combinatorial chain complex. A *combinatorial chain subcomplex* (K',∂') of (K,∂) is a chain subcomplex such that (K',∂') is itself a combinatorial chain complex with $\overline{K'} \subseteq \overline{K}$.
For notation, we simply write $(K',\partial') \subseteq (K,\partial)$.



(a) (b) (c) (d)

Figure 1.9 – Examples of three combinatorial chain complexes and a non valid one. The 0, 1, 2 dimensional cells are represented using points, lines and surfaces, respectively. They are glued together according to the boundary map. We consider complexes over $\mathbb{Z}/2\mathbb{Z}$: a q-cell appears in the boundary of a $(q+1)$-cell if they are glued together an even number of times. (a) corresponds to the chain complex of a CW-complex, and exhibits intuitive behavior. (b) and (c) exhibit exotic behavior, for instance, there are 1-cells whose boundary is equal to a single 0-cell (b) or to the sum of three distinct 0-cells (the two lower 1-cells in (c)). (d) is a subset of cells of (c) that do not form a valid combinatorial chain complex, as $\partial \circ \partial \neq 0$.

It is important to note that every subset $\overline{K'} \subseteq \overline{K}$ does not necessarily produce a valid combinatorial chain complex, as shown in (d) of Figure 1.9). A common type of combinatorial chain subcomplexes are the ones that are closed under the boundary operator:

Definition 1.3.8. Let (K', ∂') be a combinatorial chain subcomplex of (K, ∂), hence $\partial' = \mathtt{j}_{K'} \circ \partial \circ \mathtt{i}_{K'}$. We say that (K', ∂') is *closed under* ∂ if for all $x \in K'$, $\partial'(x) = \partial(x)$. Equivalently, the boundary in K of every cell of K' is a chain composed of cells of K'.

Being closed under ∂ is typically implicitly required in standard constructions of subcomplexes, such as simplicial and cellular subcomplexes, which are introduced in Section 1.3.5.

Combinatorial chain complexes provide a framework in which topology and combinatorics meet. They constitute the main objects of study throughout this work.

In the next subsection, we relate geometrical structures to this combinatorial framework. Precisely, we introduce *geometric chain complexes*, the subclass of combinatorial chain complexes that are derived from geometric complexes. In particular, we present *simplicial chain complexes*, as well as *cellular chain complexes*.

Notation. Unless specified otherwise, all chain complexes considered in the remainder of this manuscript are combinatorial chain complexes, which we simply call "complexes". In particular, when we write "let (K, ∂) be a complex", we implicitly assume that (K, ∂) is equipped with a set of cells $\overline{K}$. Similarly, a subcomplex (K', ∂') of (K, ∂) is implicitly a combinatorial chain subcomplex satisfying $\overline{K'} \subseteq \overline{K}$ and $\partial' = \mathtt{j}_{K'} \circ \partial \circ \mathtt{i}_{K'}$.

1.3.5. Geometric chain complexes

We now introduce a class of combinatorial chain complexes that arise from geometric constructions. In these complexes, cells are associated to simple geometric objects, homeomorphic to balls, and the boundary operator is induced by the geometric notion of face. An important property of these complexes is that their homology agrees with the singular homology of the underlying topological space.

To unify several commonly used constructions, we introduce the following definitions. Although they are not standard, these definitions provide an convenient framework to encompass common chain complex constructions, as they exhibit similar properties.

Definition 1.3.9 (geometric realization)**.** Let (K,∂) be a combinatorial chain complex. A *geometric realization* of (K,∂) consists of a map $|\cdot| : \overline{K} \to X$, where X is a topological space, typically $\mathbb{R}^N$ for some integer N. The realization extends naturally to subsets of cells by taking unions. In particular, we define $|K| := |\overline{K}| = \bigcup_{\tau\in\overline{K}} |\tau|$.

For the purposes of this work, geometric realizations are assumed to satisfy the following properties.

1. For every cell $\tau \in \overline{K_q}$, the set $|\tau|$ is homeomorphic to a q-dimensional ball.
2. For any two distinct cells τ and τ', the sets $|\tau|$ and $|\tau'|$ are disjoint[2].
3. For every integer q, $H_q(K) \approx H_q(|K|)$. In other words, the homology of (K,∂) coincides with the singular homology of the topological space $|K| \subseteq X$.

Definition 1.3.10 (geometric chain complex)**.** A *geometric chain complex* is a combinatorial chain complex that admits a geometric realization satisfying the assumptions of Definition 1.3.9 and that arises from a geometric complex construction. Typical examples include simplicial, cellular, cubical, polyhedral, and Δ chain complexes.

Given a geometric chain complex (K,∂), a *geometric chain subcomplex* is a combinatorial chain subcomplex that is itself induced by a geometric substructure of the underlying geometric complex. For instance, if (K,∂) is the simplicial chain complex induced by the simplicial complex X, then, a geometric chain subcomplex (K',∂') of (K,∂) is induced by a simplicial subcomplex X' of X.

Importantly, geometric chain subcomplexes inherit the geometric realization of their parent complex, which also satisfies the assumptions of Definition 1.3.9.

> *Remark.* Every combinatorial chain subcomplex of a geometric chain complex inherits a geometric realization. However, the isomorphism $H_q(K') \approx H_q(|K'|)$ does not hold for arbitrary combinatorial subcomplexes. This property is guaranteed only for geometric chain subcomplexes. See Figure 1.10 for an illustration.

The common geometric chain complexes constructions also share several additional properties.

[2]This assumption may differ from the literature, where geometric realizations of cells may intersect on lower-dimensional faces. In our work, the realization associates to each cell its **open** geometric cell.

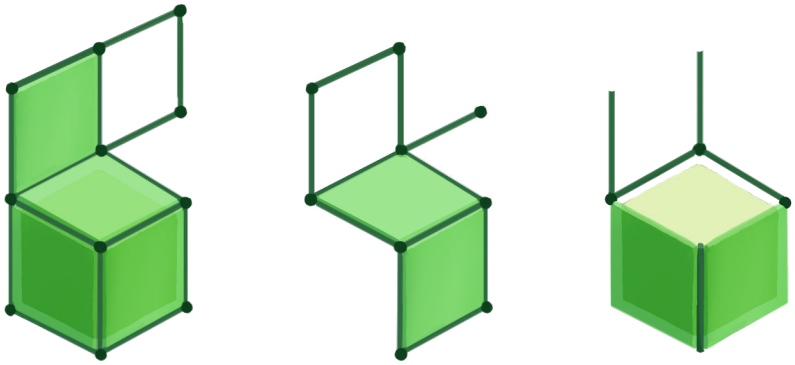


Figure 1.10 – Illustration of three combinatorial chain complexes. (K,∂) is a geometric chain complex, derived from a cubical complex. (K',∂') is a geometric chain subcomplex of (K,∂), and is thus also derived from a cubical complex. (K'',∂'') is a combinatorial chain subcomplex of (K,∂), as $\partial''\circ\partial''$ is equal to 0. However, it is not a geometric chain complex itself.

Proposition 1.3.2. *Let (K,∂) be a geometric chain complex:*

1. *if it is not empty, (K,∂) contains at least one 0-cell and satisfies $H_0(K)\neq 0$;*
2. *removing a cell of maximal dimension from (K,∂) yields a geometric chain subcomplex;*
3. *every geometric chain subcomplex (K',∂') of (K,∂) is closed under ∂ (see Definition 1.3.8).*

Moreover, when there are no multi-incidences (for instance for simplicial, cubical and polyhedral chain complexes), the converse of 3. also holds. Precisely, every combinatorial chain subcomplex that is closed under ∂ is a geometric chain subcomplex.

We now present two fundamental examples of geometric chain complexes. We first construct *simplicial chain complexes* from simplicial complexes. We then construct *cellular chain complexes* from CW-complexes.

1.3.5.1. Simplicial chain complexes

Given a simplicial complex X, we define an associated combinatorial chain complex (K,∂), called *simplicial chain complex*. For each q, we define $\overline{K_q}$ to be the set of q-simplices of X. The chain module is then defined as $K_q := \mathrm{Span}(\overline{K_q})$. Thus, $X=\overline{K}$, so the cells of (K,∂) directly correspond to the simplices of X.

The boundary operator ∂_q is defined from the facet operator δ, in analogy with the singular chain complex, see Definition 1.3.5. Precisely, for a q-simplex $[v_0,\dots,v_q]$, we have

$$\partial_q[v_0,\dots,v_q]=\sum_{i=0}^{q}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_q]$$

This definition is then extended linearly to chains. This choice of signs ensures that $\partial_q \circ \partial_{q+1} = 0$.

We now define the geometric realization of (K, ∂). For a simplex $[v_0, \dots, v_q]$, its realization is the corresponding open simplex:

$$|[v_0, \dots, v_q]| := \left\{ \sum_{i=0}^{q} t_i v_i \;\middle|\; t_i > 0 \text{ and } \sum_{i=0}^{q} t_i = 1 \right\}$$

The geometric realization of the entire simplicial complex is then defined as $|K| = \bigsqcup_{\tau \in \overline{K}} |\tau|$.

A standard result is that the homology of the simplicial chain complex are isomorphic to the singular homology its geometric realization, that is, $H_q(K) \approx H_q(|K|)$. For a proof of this result, see HATCHER 2002, Chapter 2. All in all, this definition satisfies the assumptions of Definition 1.3.9, which ensures that simplicial chain complexes can be considered as geometric chain complexes.

A simplicial subcomplex X' of X induces a simplicial chain complex (K', ∂') that is a combinatorial chain subcomplex of (K, ∂). However, as mentioned above, general combinatorial chain subcomplexes of (K, ∂) may not correspond to simplicial subcomplexes of X.

1.3.5.2. Cellular chain complexes

Given a CW-complex X, we define its associated combinatorial chain complex (K, ∂), called *cellular chain complex*. Its construction relies on singular homology, relative homology, and the degree of continuous maps. We refer the reader to HATCHER 2002 for a detailed explanation. Here, we only briefly present the main ideas.

For each dimension q, the set of q-cells $\overline{K_q}$ is identified with the set $\mathscr{C}_q$ of q-cells of the CW-complex. The boundary operator is defined from the gluing maps of the CW-complex, together with the notion of *degree*. Intuitively, the boundary of a q-cell is the sum of the $(q-1)$-cells on which it is attached, with coefficients depending on how many times, and with which orientation, the gluing map wraps around each $(q-1)$-cell.

The geometric realization of a CW-complex is given by the space itself. Precisely, the realization of a cell τ is $|\tau| := B_\tau \subseteq X$, therefore $|K| = X$. A fundamental result is that the homology of (K, ∂) is isomorphic to the singular homology of the space X. Thus, cellular chain complexes satisfy the assumptions of Definition 1.3.9, and therefore can be considered as geometric chain complexes.

Every CW-subcomplex X' of X induces a cellular chain complex (K', ∂') that is a combinatorial chain subcomplex of (K, ∂). Conversely, not every combinatorial chain subcomplex of a cellular chain complex is induced by a CW-subcomplex. See Figure 1.10 for an illustration.

1.3.5.3. Other geometric chain complexes

It is possible to also define geometric chain complexes from other geometric complex constructions. A standard construction is *cubical chain complexes*, derived from cubical complexes. Another example is Δ *chain complexes*, built from Δ-complexes.

Regarding polyhedral complexes, it can be more challenging to build a proper boundary operator. A common idea is to triangulate the polyhedral complex and infer the boundary from the resulting simplicial chain complex. However, this approach is usually computationally expensive. ALAYRANGUES, PELTIER, et al. 2009; ALAYRANGUES, DAMIAND, et al. 2015 propose alternative approaches to compute a boundary operator (and so a combinatorial chain complex) for combinatorial and generalized maps.

In all cases, these constructions (when using a finite number of cells) can be considered as subclasses of geometric chain complexes.

Notation. Throughout our work, we identify a geometric complex with its associated geometric chain complex, when no confusion can be made. In particular, we write sentences like "let (K,∂) be a geometric complex" to implicitly define (K,∂) to be the geometric chain complex of a defined geometric complex. Similarly, "let (K,∂) be a simplicial complex" implicitly defines $\overline{K}$ to be a simplicial complex, (K,∂) its corresponding simplicial chain complex, and $|\cdot|$ the associated geometric realization.

1.3.6. Intuition, holes and homology bases

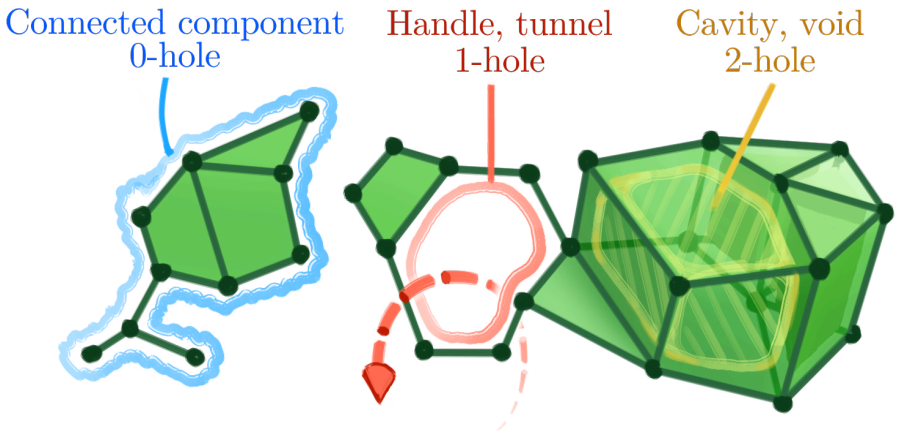


Figure 1.11 – Illustration of the intuition behind homology for geometric complexes. Note that this intuition fails in general for general combinatorial chain complexes.

When considering common geometric complexes, the homology groups are intuitively related to the holes of the underlying topological space. Depending on q, the q-homology groups represents a different feature, that is counted by the Betti number β_q. For instance, in 3D, β_0 gives the number of connected components (0-holes), β_1 the number of tunnels (1-holes) and β_2 the number of cavities (2-holes), see Figure 1.11.

A way to intuitively visualize these holes is with the *homology bases* and *generators*. They provide relevant algebraic representations of homological holes that can also carry both a combinatorial and geometrical meaning, as explained in Chapters 2 and 4.

Definition 1.3.11 (homology bases and generators)**.** A family $\{g_i\}_{i=1}^{\beta_q}$ of q-cycles is a *q-homology basis* of a chain complex (K,∂) if and only if $\left\{[g_i]_q^K\right\}_{i=1}^{\beta_q}$ is a basis of $H_q(K)$[3]. The cycles g_i are called *q-homology generators*. See Figure 1.12 for an illustration.

Intuitively, the homology generators correspond to closed curves (for dimension 1) or closed surfaces (for dimension 2) surrounding the holes.

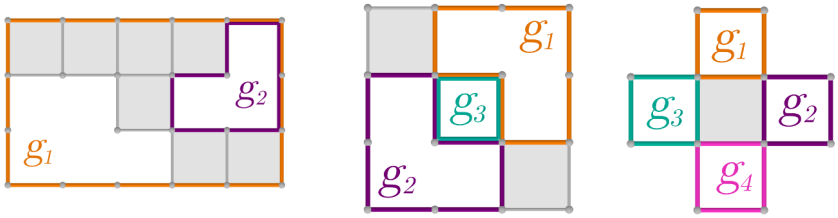


Figure 1.12 – Examples of three 1-homology bases for three different complexes.

This intuition is an important aspect of our work, as we are interested in recovering and computing it. For instance, in Chapter 4, we define four geometric representations of homology elements that are related to the intuition of holes. Precisely, we define chains that intuitively surround, passe through, fill, and cut the holes. In addition, in Chapter 5 we compute the *thickness* and the *breadth* of holes obtained from homology, and a current work is to define a *shape* of a homological hole.

1.3.7. Cohomology

We now introduce the dual notion of homology, namely, cohomology. Cohomology can be understood as the homology we obtain when we reverse the boundary arrows in the chain complexes.

Definition 1.3.12 (dual complex)**.** Given a chain complex (K,∂), its dual complex is the chain complex $(K,\partial)^*$ whose chains are the same but in reverse order, and equipped with the *coboundary* operator ∂^*, which is the transpose of ∂:

$$\{0\} \xleftarrow{\partial^*_{n+1}} K_n \xleftarrow{\partial^*_n} \dots \xleftarrow{\partial^*_{q+1}} K_q \xleftarrow{\partial^*_q} K_{q-1} \xleftarrow{\partial^*_{q-1}} \dots \xleftarrow{\partial^*_1} K_0$$

Elements in $\ker\partial^*$ and $\operatorname{im}\partial^*$ are respectively called *cocycles* and *coboundaries*.

[3] Precisely, it is a basis of the free part of $H_q(K)$. By abuse of notation, we consider it is a basis of $H_q(K)$, as we mainly work over a field, and the torsion part is null in this case.

When (K,∂) is a combinatorial chain complex, the q-cells of (K,∂) are considered as the $(n-q)$-cells of $(K,\partial)^*$: $\overline{K_q} = \overline{K^*_{n-q}}$. See Figure 1.13 for an illustration of the dual complex.

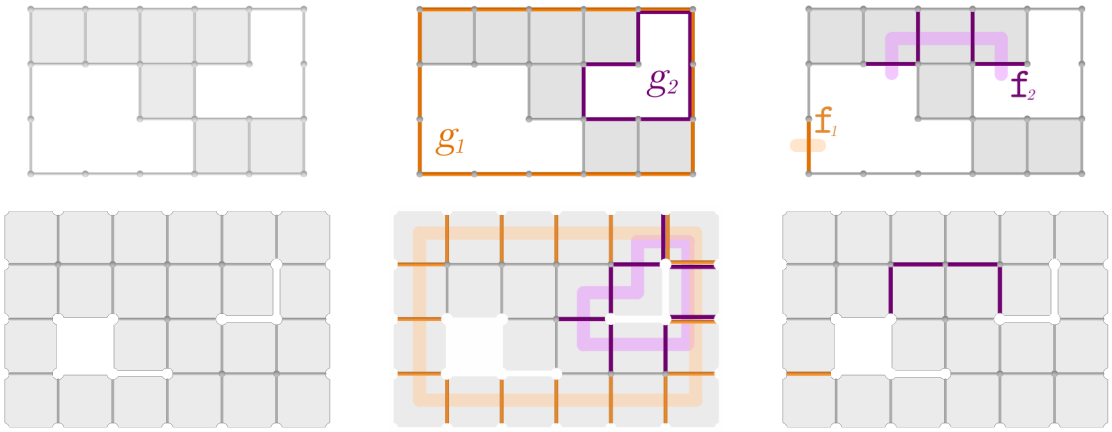


Figure 1.13 – The first row shows a complex (K,∂) together with a 1-homology basis $\{\mathtt{g}_1,\mathtt{g}_2\}$ (middle) and a 1-cohomology basis $\{\mathtt{f}_1,\mathtt{f}_2\}$ (right). The second row represents the corresponding dual objects: the dual complex $(K,\partial)^*$ (left), the duals of $\{\mathtt{g}_1,\mathtt{g}_2\}$ (middle) and $\{\mathtt{f}_1,\mathtt{f}_2\}$ (right), which respectively are 1-cohomology and 1-homology bases for $(K,\partial)^*$. For the dual complex $(K,\partial)^*$, every cell of $\overline{K_q} = \overline{K^*_{2-q}}$ is visually represented with a transversal cell of dimension $2-q$.

The q-cohomology group of (K,∂) is then defined as the $(n-q)$-homology group of $(K,\partial)^*$:

$$H^q\big((K,\partial)\big) = H_{n-q}\big((K,\partial)^*\big) = \ker\partial^*_{q+1} / \operatorname{im}\partial^*_q$$

This implies that a homology basis for $(K,\partial)^*$ is a cohomology basis for (K,∂) and vice versa. In the remaining of this manuscript, to improve clarity, cohomology generators are also represented by a dual path (with a lighter color) intuitively cutting through the holes, see Figure 1.13.

If (K,∂) is finitely generated and without torsion, $H_q(K)$ is isomorphic to $H^q(K)$. In particular, $\dim(H_q(K)) = \beta_q = \dim(H^q(K))$. When there is torsion, we can write $H_q(K) \approx \mathscr{R}^{\beta_q} \oplus T_q$ and $H_{q-1}(K) \approx \mathscr{R}^{\beta_{q-1}} \oplus T_{q-1}$, and we have $H^q(K) \approx \mathscr{R}^{\beta_q} \oplus T_{q-1}$, see Hatcher 2002(corollary 3.3).

In general, cohomology gives as much information about the topology as homology, which feels natural as the two concepts are dual of each other. This is in particular the case for the results presented in the following chapters. However, counter-intuitively, cohomology can be equipped with a product $\smile$: $H^p(K) \times H^q(K) \to H^{p+q}$, called *cup product*, which can be used to get some extra topological information. For instance, Mémoli et al. 2024; Ivshina et al. 2025; Lupo et al. 2022 used the cup product to differentiate homologous but non homeomorphic spaces, like a torus and two circles and one sphere glued together.

Cohomology also allows the statement of powerful duality results. For instance, the Poincaré duality states that if X is an oriented closed n-manifold, then $H^q(K)$ is isomorphic to $H_{n-q}(X)$, for all q. Another duality result is Alexander duality, which is introduced in Chapter 4, and also used in Chapter 5.

1.3.8. A word about homotopy

We briefly introduce homotopy for completeness, as we sometimes use homotopy arguments and deformation retracts in our proofs. However, keep in mind that homotopy is only a side character in our work.

Given two topological spaces X and Y, a *homotopy* between continuous functions $f_0 : X \to Y$ and $f_1 : X \to Y$ is a continuous function $H : [0,1] \times X \to Y$, such that $H(0,\cdot) = f_0$ and $H(1,\cdot) = f_1$. H can be understood as a continuous deformation of f_0 into f_1. A *homotopy equivalence* between X and Y is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$, such that there exists a homotopy between $g \circ f$ and id_X, and between $f \circ g$ and id_Y. If such a pair exists, X and Y are said to be homotopy equivalent. In particular, this implies that X and Y have isomorphic singular homology groups for every dimension.

We present here a special case of homotopy equivalence, which is used in some of the proofs of this manuscript.

Definition 1.3.13. A *deformation retract* from a space X to a subspace $Y \subseteq X$ is a continuous map $H : [0,1] \times X \to X$, such that:

$$H(0,\cdot) = id \qquad \forall x \in X \;\; H(1,x) \in Y \qquad \forall y \in Y \;\; H(1,y) = y$$

If such a deformation retract exists, X and Y are homotopy equivalent.

Another approach to homotopy is via the definition of homotopy groups. The q-th homotopy group $\pi_q(X)$ of X is the set of homotopy classes of continuous functions from the q-sphere into X. For instance, $\pi_1(X)$ intuitively classifies the loops on X, up to continuous deformation.

It has been shown that homotopy groups are strictly stronger invariants than homology groups, that is, $\pi_q(X) \approx \pi_q(Y) \implies H_q(X) \approx H_q(Y)$. However, due to their non-commutative nature, the homotopy groups are harder to compute. Indeed, deciding if two finite CW-complexes have isomorphic homotopy groups is undecidable by Turing machine, whereas it is decidable in polynomial time for homology groups (see for instance RUBIO et al. 2002).

1.4. Computing Homology

Homology can be computed whenever the complex under study is finite, as in the case of combinatorial chain complexes. Several approaches have been developed to compute different types of homological information. These approaches constitute

the field of computational homology. Given a combinatorial chain complex (K, ∂) and a fixed ring $\mathscr{R}$, we introduce a hierarchy of homological information, as different methods compute different levels of information. Every level can be deduced from the higher ones.

- **Level 0.** The Euler characteristic of K.
- **Level 1.** The Betti numbers of K.
- **Level 2.** A decomposition of the homology groups, including torsion when working over a ring.
- **Level 3.** Homology generators and bases of K.
- **Level 4.** Homology groups and bases, together with decompositions of cycles.

The Euler characteristic can be computed in $O(N)$ time, with N being the number of cells of K, as it is given by the alternating sum of the number of cells in each dimension.

Starting from Level 1, the computational complexity is generally cubic[4] in N. However, for low-dimensional complexes, faster algorithms exist. For instance, DELFINADO et al. 1995 propose a quasi-linear time algorithm for computing Betti numbers of a simplicial subcomplex of the 3-sphere. Similarly, GONZALEZ-LORENZO, JUDA, et al. 2016 introduce a linear-time algorithm for the Betti numbers of 3D cubical complexes.

For higher levels of information, several types of approaches can be considered. The traditional way of computing homology groups relies on the computation of the *Smith normal form* (SNF) of the boundary matrices. In its standard form, this method provides Level 2 information, but can be slightly modified to compute level 3 information. This approach is discussed in Section 1.4.1.

Another approach is based on the concept of *reduction*, which aims to "simplify" a large complex into a smaller one while preserving its homology. This concept emerged from the theory of *effective homology*, introduced by RUBIO et al. 1986, which studies the computability of algebraic topology concepts. When the reduction is *perfect*, this approach computes Level 4 information. It is presented in Section 1.4.2.

Finally, *discrete Morse theory*, introduced by FORMAN 1998, generalizes Morse theory to the discrete setting. In this theory, homological information is computed using *discrete gradient vector fields* defined on the complex. This approach also relies on effective homology and yields Level 4 information when no obstruction occurs. It is presented in Section 1.4.3.

After presenting these three standard approaches, we dedicate a whole section to persistent homology, a central object in computational homology and topological data analysis, see Section 1.5. Persistent homology is usually computed over a field and provides Level 1 or Level 2 information. Moreover, it also provides information of other nature, encoded in a *persistence diagram*.

To conclude this first chapter, Section 1.6 introduces *homological discrete vector fields* (HDVF), which constitute the central object of study and tool in the following chapters. HDVFs generalize discrete Morse theory and also rely on effective homology.

[4] Matrix multiplication time algorithms are theoretically possible, see MILOSAVLJEVIĆ et al. 2011

They compute Level 4 information when working over a field. Moreover, they provide a combinatorial framework that unifies and encodes the previous methods, making them a relevant tool for studying the computations of homology in general.

1.4.1. The Smith normal form

The boundary operator of a chain complex is linear. Therefore, for a combinatorial chain complex (K, ∂), each ∂_q can be represented by a matrix, in the basis of q-cells.

The traditional method for computing homology groups consists in computing the *Smith normal form* of these boundary matrices. See MUNKRES et al. 2025 for a detailed exposition.

Definition 1.4.1 (Smith normal form)**.** Let A be a matrix of $M_{n\times m}(\mathscr{R})$. The *Smith normal form* (SNF) of A is the following matrix $D \in M_{n\times m}(\mathscr{R})$:

$$D = \begin{bmatrix} id_s & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{with} \quad \Delta = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \alpha_r \end{bmatrix}$$

such that α_i divides α_{i+1} for all i, and such that there exist invertible matrices P and Q satisfying $A = PDQ$.

From the Smith normal form of the q-th boundary matrix ∂_q, we can directly read the rank of q-homology group, hence the q-th Betti number.

Briefly, suppose that for all q, ∂_q is a matrix of size $N_q \times N_{q-1}$, with N_q the number of q-cells. The Smith normal form of ∂_{q+1} and ∂_q produces:

$$D_{q+1} = \begin{bmatrix} id_{s_{q+1}} & 0 & 0 \\ 0 & \operatorname{diag}\left(\alpha_1^{q+1}, \ldots, \alpha_{r_{q+1}}^{q+1}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad D_q = \begin{bmatrix} id_{s_q} & 0 & 0 \\ 0 & \operatorname{diag}\left(\alpha_1^{q}, \ldots, \alpha_{r_q}^{q}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of $\ker \partial_q$ is given by $N_q - s_q - r_q$. The rank of $\operatorname{im} \partial_{q+1}$ is $s_{q+1} + r_{q+1}$. As $H_q(K) = \ker \partial_q / \operatorname{im} \partial_{q+1}$, separating the free and torsion part yields the following:

$$\beta_q(K) = N_q - s_q - r_q - s_{q+1} \qquad T_q \approx \mathscr{R}/\alpha_1^{q+1}\mathscr{R} \oplus \cdots \oplus \mathscr{R}/\alpha_{r_{q+1}}^{q+1}\mathscr{R}$$

Computing the SNF can be achieved by an algorithm similar to Gaussian elimination, with the constraint that each pivot must divide all other entries. When working over a field, this simply corresponds to standard Gaussian elimination. Note that, for homology over a $\mathbb{Z}$, this computation may produce very large integers, see for instance PELTIER et al. 2006.

Importantly, this method does not straightforwardly provide homology and cohomology bases. Indeed, the choices of pivots, during the matrix reductions of boundary

matrices, may lead to incompatible chain bases in between dimensions. This prevents from extracting a homology basis from this computation.

To address this issue, it has been proven that one can perform simultaneous reductions of ∂_{q+1} and ∂_q using the same chain bases. This leads to a similar form, known as the *modified Smith normal form* (mSNF), see PELTIER et al. 2006. In this form, homology bases can be extracted, even in the presence of torsion.

1.4.2. Reductions and effective homology

When a chain complex is too large for a direct SNF computation, an alternative approach is to use *reductions*. A reduction links two chain complexes, typically a large one and a smaller one, while preserving their homology. This notion emerged from the theory of *effective homology*, introduced in RUBIO et al. 1986, which studies the computability of algebraic topology concepts. In particular, this theory investigates the computation the homology of topological spaces where only local information is available (e.g. neighborhood of cells). Similar approaches for reducing the size of complexes have also been proposed, see for instance KACZYŃSKI et al. 1998; MROZEK et al. 2009; DŁOTKO et al. 2011.

Definition 1.4.2 (reduction RUBIO et al. 1986)**.** A *reduction* from a chain complex (K,∂) to a chain complex (K',d) is a triplet of maps (f,g,h) with $f_q : K_q \to K'_q$, $\quad g_q : K'_q \to K_q$ and $h_q : K_q \to K_{q+1}$ satisfying the following properties:

1. f and g are chain maps, i.e. $f\partial = df$ and $\partial g = gd$;
2. $fh = 0$, $hg = 0$, and $hh = 0$;
3. $fg = id_{K'}$;
4. $gf = id_K - \partial h - h\partial$.

$$\begin{array}{ccccccccc}
\cdots & \underset{h_q}{\overset{\partial_{q+1}}{\rightleftarrows}} & K_q & \underset{h_{q-1}}{\overset{\partial_q}{\rightleftarrows}} & K_{q-1} & \underset{h_{q-2}}{\overset{\partial_{q-1}}{\rightleftarrows}} & \cdots \\
 & & \Big\uparrow{\scriptstyle g_q} \Big\downarrow{\scriptstyle f_q} & & \Big\uparrow{\scriptstyle g_{q-1}} \Big\downarrow{\scriptstyle f_{q-1}} & & \\
\cdots & \xrightarrow{d_{q+1}} & K'_q & \xrightarrow{d_q} & K'_{q-1} & \xrightarrow{d_{q-1}} & \cdots
\end{array}$$

The existence of a reduction implies that, for all q, $H_q(K) \approx H_q(K')$. An explicit example of a reduction is given later in Section 1.6.

Definition 1.4.3 (perfect reduction RUBIO et al. 1986)**.** A reduction (f,g,h) from (K,∂) to (K',d) is *perfect* if $d = 0$. In this case, for all q, we have $H_q(K) \approx K'_q$.

In addition, when (K,∂) is a combinatorial chain complex, $\{g_q(\gamma)\}_{\gamma \in \overline{K'_q}}$ is a q-homology basis and $\{f^*_q(\gamma)\}_{\gamma \in \overline{K'_q}}$ is a q-cohomology basis.

Perfect reductions satisfy several useful properties that we use extensively in our proofs. For instance, since $\partial g = 0$, every element of the form $g(x)$ is a cycle. Since $g^*\partial^* = 0$, the map g^* sends coboundaries to zero. Note that g and f^* plays a dual role, in the sense of the duality homology-cohomology. Thus, we also have that every $f^*(x)$ is a cocycle and f sends boundaries to zero.

Moreover, using 4. of Definition 1.4.2, a reduction induces a decomposition of cycles. Indeed, given a cycle x we have $x = gf(x) + \partial h(x)$, with $gf(x)$ being a representative of the homology class $[x]$ and $\partial h(x)$ being a boundary.

In the literature, reductions are also referred to as *contractions* (EILENBERG et al. 1953), *chain contractions* (GONZÁLEZ-DÍAZ et al. 2009), or *trivialized extensions* (MUNKHOLM 1974). The map h is usually interpreted as a chain homotopy map.

Besides the algebraic definition above, reductions have also been formalized in more combinatorial settings, for instance with AT-models (REAL et al. 2009). They have proved to be useful both in the study of complicated spaces, such as loop spaces, and in applications including image analysis, see for example GONZALEZ-DIAZ, Maria Jose JIMENEZ, et al. 2011.

1.4.3. Discrete Morse theory and DGVFs

Differential topology aims to obtain topological information about a space by studying differentiable functions defined on it. A classical result of this theory is the Reeb sphere theorem, which states that a compact differentiable manifold admits a smooth function with exactly two non-degenerate critical points if and only if it is homeomorphic to an n-sphere.

In a similar vein, *Morse theory* relates the topology of a smooth manifold to the critical points (assumed non-degenerate) of smooth functions defined on it. Discrete Morse theory, introduced in the late 90's by FORMAN 1998; FORMAN 2001, provides a discrete analogue for CW-complexes. Its main object is the *discrete gradient vector field*, defined from a matching on the *Hasse diagram* of the complex.

Definition 1.4.4. The *Hasse diagram* of a CW-complex (K, ∂) is the directed graph whose vertices are the cells of K, and where there is an edge from τ to τ' whenever τ' is a facet of τ. This graph is acyclic and is denoted by $\mathscr{H}^K$. We also denote by $\mathscr{H}^K_q$ the graph $\mathscr{H}^K$ restricted to the cells of dimensions q and $q+1$.

Definition 1.4.5. A *discrete gradient vector field* (DGVF) on (K, ∂) is a collection of edges X of $\mathscr{H}^K$ such that:

1. no two edges in X share a common vertex;
2. reversing the edges in X does not create any directed cycle in $\mathscr{H}^K_q$, for all q.

The graph obtained after reversing the arrows is called the *Morse graph* of X. DGVFs are also referred to as *acyclic matchings*, see for instance HARKER et al. 2014 They are also related to *Morse sequences* introduced in BERTRAND 2025. See Figure 1.14 for an example.

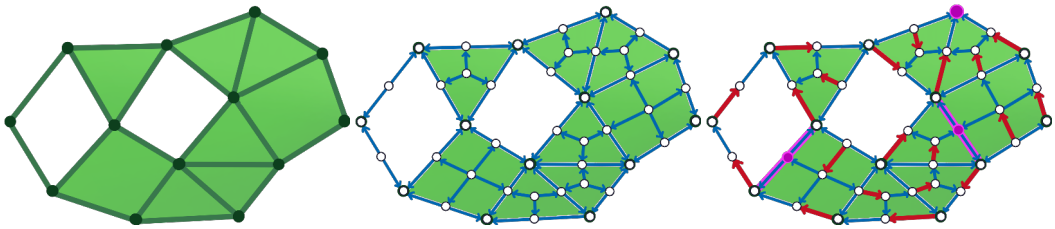


Figure 1.14 – A complex (left), its Hasse diagram (middle) and the Morse graph of a perfect DGVF (right). Reversed edges are shown in red, and cells not incident to any reversed edge (i.e. critical cells) are colored in pink.

Given a DGVF, the cells that do not belong to any reverted arrows are called *critical*. DGVFs actually encode sequences of elementary collapses, and thus induce an associated reduction from the original complex to a smaller complex generated by the critical cells. The map h is then expressed as an infinite sum over a nilpotent expression (the nilpotency is actually guaranteed by the acyclicity assumption). For a detailed construction of this reduction, refer to GONZALEZ-LORENZO, BAC, MARI, et al. 2017 or to BERTRAND 2025, where it is described using the notions of *(co)reference* maps.

When the reduction is perfect, the DGVF is said to be perfect, and the number of critical q-cells equals the q-th Betti number. In general, however, a DGVF only provides an upper bound on the Betti numbers.

Building a perfect DGVF is often challenging. Indeed, greedily adding arrows to a DGVF may lead to a dead end, even when alternative choices would give a better or perfect DGVF. This raises the question of whether a CW-complex admits a perfect DGVF. In general, the answer is negative. Indeed, a DGVF encodes a sequence of elementary collapses. Hence, any contractible but non-collapsible complex (e.g. the dunce hat or Bing's house) does not admit a perfect DGVF. This obstruction motivates the introduction of *homological discrete vector fields*, which relax the acyclicity constraint and always allow the greedy computation of perfect reductions, see Section 1.6.

Discrete Morse theory is also closely related to persistent homology. Recent works has explored how to extend it to more general persistence frameworks. These include extension to zigzag persistence (MARIA et al. 2024), as well as to multiparameter persistence (LANDI et al. 2022; ALLILI et al. 2019; BROUILLETTE et al. 2024).

1.5. Persistent Homology

In the 90's, persistent homology emerged independently through the work of EDELSBRUNNER et al. 2002 (North Carolina), FROSINI 1992 (Bologna), and ROBINS 1999 (Colorado), extending the study of homology to increasing sequences of complexes.

This later theory proved to have a wide range of applications in topological data analysis and hence quickly developed and spread in the scientific community. It is now widely used in data science. For instance, persistent homology can be incorporated as

a feature in machine learning models and neural networks, see for instance CARRIÈRE et al. 2020, or PUN et al. 2022 for a comprehensive survey. Other methods use persistent homology for dimensionality reduction of high-dimensional data, see CLÉMOT, DIGNE, and TIERNY 2025; DORAISWAMY et al. 2020. Applications in other fields include biological data analysis (María José JIMENEZ et al. 2017), mathematical morphology and image analysis (PASSAT et al. 2023; BOUTRY et al. 2022), neuroscience (CURTO et al. 2025), sketch contour completion (KURLIN 2014), and music analysis (M. BERGOMI et al. 2020).

The strength of persistent homology relies on two key properties. First, it provides a notion of "importance" for homology artifacts, allowing to distinguish meaningful holes from noise. Second, it is stable under perturbations of the input data, which makes it a robust tool for analyzing real-world data.

1.5.1. Definition

Persistent homology studies the evolution of homology groups in a *filtration*. Formally, a filtration is a growing sequence of chain complexes $K^0 \subseteq K^1 \subseteq \cdots \subseteq K^N$. The inclusion maps between complexes induce maps between their homology groups. For each dimension q we have:

$$H_q(K^0) \xrightarrow{\iota_q^0} H_q(K^1) \xrightarrow{\iota_q^1} \dots \xrightarrow{\iota_q^{N-1}} H_q(K^N)$$

Given $i < j$, the image of $\iota_q^i \circ \cdots \circ \iota_q^{j-1}$ represents the q-homology classes of K^i that persist until K^j. Intuitively, these correspond to q-holes of K^i that are not filled between steps i and j. It is then possible to define a *birth* and a *death* index for each homology class that appears in the filtration. Precisely, a q-homology class is born at index i if it belongs to $H_q(K^i) \setminus \mathrm{im}(\iota_q^{i-1})$. It dies at index j if it belongs to $\ker(\iota_q^i \circ \cdots \circ \iota_q^{j-1})$ but not to $\ker(\iota_q^i \circ \cdots \circ \iota_q^{j-2})$.

Thus, every homology class in the filtration is associated with a lifetime interval $[i, j]$, where j may be infinite if the class never dies. These intervals are summarized in a *persistence diagram*, which can be defined as a multiset of points in $\mathbb{R}^2$. See H. EDELSBRUNNER and HARER 2008; ZOMORODIAN et al. 2005 for a detailed presentation of persistent homology. See Figure 1.15 for an illustration of a filtration and its persistence diagram.

Persistent homology can also be defined with other formalisms. These include a categorical approach BUBENIK and SCOTT 2014, an algebraic approach using modules over polynomial rings ZOMORODIAN et al. 2005, and an approach using quiver representations OUDOT 2015.

> *Remark.* The framework of persistent homology also works in the continuous setting. In this case, a filtration consists of a growing sequence of topological spaces, and we consider singular homology.

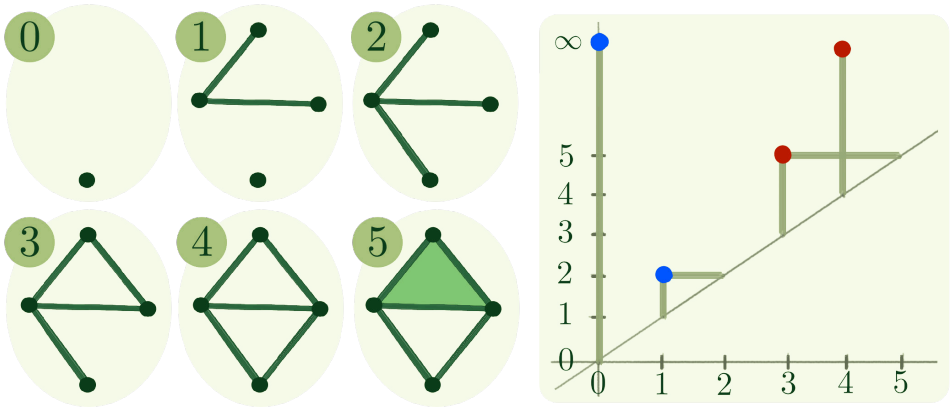


Figure 1.15 – A filtration and its associated persistence diagram. Blue and red points correspond to homology classes of dimensions 0 and 1, respectively, that appear and eventually disappear during the filtration.

1.5.2. Stability

A key advantage of persistent homology is its stability with respect to perturbations of the input data. A small perturbation in the input filtration results in a small change in the persistence diagram. The classical stability theorem for persistent homology was established in COHEN-STEINER, H. EDELSBRUNNER, and HARER 2005 in the continuous setting.

Theorem 1.5.1 (stability of persistent homology, COHEN-STEINER, H. EDELSBRUNNER, and HARER 2005)**.** *Let X be a triangulable space, and let $f, g : X \to \mathbb{R}$ be continuous tame functions. Then,*

$$d_B(Dgm(f), Dgm(g)) \leq ||f - g||_\infty$$

where d_B is the bottleneck distance (or matching distance) and $Dgm(f)$ and $Dgm(g)$ are the persistence diagrams obtained by the filtrations induced by f and g.

This result was later generalized in CHAZAL, COHEN-STEINER, GLISSE, et al. 2009 by removing the triangulable and continuous assumptions, and by weakening the tameness condition. It was also extended to Wasserstein distance in the case of Lipschitz functions in COHEN-STEINER, H. EDELSBRUNNER, HARER, and MILEYKO 2010.

Stability can also be formulated in algebraic and categorical settings, using for example the notion of *interleaving distance* (or *natural pseudo distance* in D'AMICO et al. 2008), see for instance PATEL 2018; LESNICK 2015; CHAZAL, COHEN-STEINER, GLISSE, et al. 2009 for more details.

1.5.3. Computing persistent homology

The computation of persistent homology can be performed using different approaches. In practice, computations are mostly done with coefficients in the field $\mathbb{Z}/2\mathbb{Z}$.

Common approaches relies on matrix reduction of the boundary matrices. They are closely related to Smith normal form methods, but are adapted to filtrations. Such methods include EDELSBRUNNER et al. 2002; ZOMORODIAN et al. 2005.

Another approach is via reduction techniques, for instance using elementary collapses or discrete Morse theory. Such methods include MISCHAIKOW et al. 2013; WILKERSON et al. 2014.

Other non common approaches usually incorporate combinatorial optimizations and heuristics to speed up computations in practice, see for instance CHEN et al. 2011.

The worst-case time complexity of most methods is cubic in the number of cells in the filtration. Though, it is faster in practice, for instance when the boundary matrix is sparse. It is also known that persistent homology can be computed in matrix multiplication time, see MILOSAVLJEVIĆ et al. 2011, but such approaches are not used in practice. For a comprehensive survey about the computation of persistent homology, please refer to OTTER et al. 2015, which also provides benchmarks of persistent homology softwares.

Later, in Section 3.2, we present the standard abstract persistent homology algorithm introduced by EDELSBRUNNER et al. 2002.

1.5.4. Extensions of persistent homology

Persistent homology provides useful information in many contexts. However, its main restriction is that its input is a filtration of topological space, which may not correspond to how data are naturally encoded. For this reason, persistent homology has been extended in several ways, in order to provide a useful analysis tool for wider contexts.

A natural extension of persistent homology is to ease the restriction that a filtration is an increasing sequence of objects. One such extension is multidimensional persistent homology (G. CARLSSON et al. 2007; BIASOTTI et al. 2008), which aims to generalize persistent homology to multidimensional filtrations and is currently a fruitful topic with applications in topological data analysis. Another is zigzag persistence (G. E. CARLSSON et al. 2008), where the maps between two consecutive spaces in the filtration can point in either direction.

A theoretical generalization of persistent homology is to develop the concept of persistence itself. This induced the generalization of persistence using category theory (BUBENIK and SCOTT 2014; BUBENIK, SILVA, et al. 2014; PATEL 2018). Within this new paradigm, concepts related to homology have been replaced by other abstract categorical concepts, and the focus has moved from homology to the persistence function itself. From this point of view, persistence is seen as a process that takes a filtration and produces a persistence diagram by analyzing how a specific concept persists along the filtration.

Another extension was driven by the will to use a persistence diagram on objects other than topological sets and for features other than homology, typically to study certain properties of graphs. An approach is to build a topological complex whose homology corresponds to the characteristics studied (see, for example, GRIGOR'YAN,

MURANOV, and JIMÉNEZ 2021; GRIGOR'YAN, LIN, et al. 2020; GRIGOR'YAN, MURANOV, and YAU 2014; BRESSAN et al. 2016). Another approach is to use the aforementioned categorical persistence framework to study persistent features without using homology and topological objects. An example is the study of persistent hypergraph attributes in LIU et al. 2021; LIU et al. 2024 and in BABU et al. 2024. Rank-based persistence (M. G. BERGOMI and VERTECHI 2020) and graph persistence (M. G. BERGOMI and FERRI 2023) were also developed in this context, as well as steady and ranging persistence (M. G. BERGOMI, FERRI, and TAVAGLIONE 2023), which has been the topic of a work made in parallel of this PhD thesis (see GAZULL 2025).

1.6. Homological Discrete Vector Fields

In this section, we present the *homological discrete vector field* (abbreviated as *HDVF*), which is both an object of study and a tool throughout our work.

Historically, HDVFs emerged as a generalization of Discrete Morse Theory FORMAN 1998. Precisely, it was shown that DGVFs sometimes fails to compute homology, for instance for the Bing's house. In the 2D planar setting (in the context of images for instance), this limitation does not occur, and works such as MOLINA-ABRIL et al. 2012a; MOLINA-ABRIL et al. 2012b about homological spanning forests highlight the relevance of such combinatorial approaches. However, they could not be extended to higher dimensions. HDVFs were first developed to overcome this limit, by relaxing the acyclicity constraint of the Morse diagram. The key idea was to allow cyclic morse diagrams that still produce valid reductions. A central observation is that, to construct the associated reduction, the specific pairing of cells in a DGVF is less important than the partition of cells it induces.

Coefficient choice. In the remainder of this manuscript, we work with coefficients in a fixed **field** $\mathscr{R}$. In this case, chain groups are actually vector spaces. For examples and illustrations, we consider the field $\mathbb{Z}/2\mathbb{Z}$, so that chains can intuitively be identified with sets of cells.

Working with a general ring introduces complications, which constitute a major challenge. Yet, in this section, most definitions and results are still valid. When working over ring, the main limitation is that greedy completions of HDVF may fail to produce a perfect HDVF. More precisely, in this section, the propositions that are not valid in a general ring are Proposition 1.6.4 and the completion results of Section 1.6.3.

Notations. We work exclusively with combinatorial chain complexes, which we simply refer to as *complexes*. In particular, our formalism applies to geometric chain complexes.

We now recall and clarify some notations. Let (K,∂) be a complex. Given a subspace $E \subseteq K$, we denote by $\mathtt{i}_E$ is the inclusion map from the subspace E into K, $\mathtt{j}_E$ the corresponding projection map from K to E and id_E the identity map on E. Now, given a set of cells $A \subseteq \overline{K}$, we also write $\mathtt{i}_A := \mathtt{i}_{\operatorname{Span} A}$, $\mathtt{j}_A := \mathtt{j}_{\operatorname{Span} A}$ and $id_A := id_{\operatorname{Span} A}$.

Moreover, given two sets of cells A and B, we denote $\partial_A^B := \mathtt{j}_B \circ \partial \circ \mathtt{i}_A$ the boundary induced by Span A restricted to Span B.

Recall also that we write $(K',\partial') \subseteq (K,\partial)$ to mean that (K',∂') is a combinatorial chain subcomplex of (K,∂) i.e. it is the complex generated by the cells $\overline{K'} \subseteq \overline{K}$ with the boundary map given by $\partial' := \mathtt{j}_{K'} \circ \partial \circ \mathtt{i}_{K'}$. For simplicity, we often identify ∂ and ∂' and write (K',∂) instead of (K',∂') when no confusion arises.

1.6.1. Definition and main properties

A HDVF is a combinatorial object that computes homological information of combinatorial chain complexes. By nature, it intertwines the notions of cells and chains. For this reason, it is important in the following to understand the subtleties between these two notions.

Definition 1.6.1 (Gonzalez-Lorenzo, Bac, Mari, et al. 2017)**.** Given a complex (K,∂), a *homological discrete vector field* is a triplet $X = (P,S,C)$ with P, S, and C forming a partition of $\overline{K}$ such that ∂_S^P is invertible. The cells of P, S and C are respectively called *primary*, *secondary* and *critical* cells. In figures, they are respectively colorized in green, red and blue, see Figure 1.16.

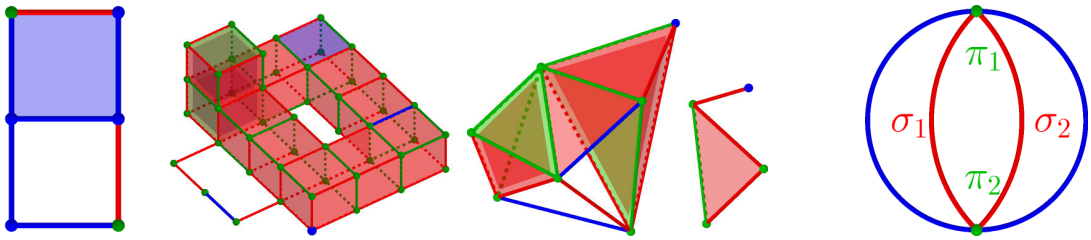


Figure 1.16 – Four partitions of the cells of complexes into primary (in green), secondary (in red) and critical (in blue) cells. The first three partitions are HDVFs whereas the last one is not a HDVF: ∂_S^P is not invertible because $\partial_S^P(\sigma_1) = \partial_S^P(\sigma_2) = \pi_1 + \pi_2$.

As for discrete Morse theory, every HDVF induces a reduction from the initial complex to a complex generated by the critical cells.

Proposition 1.6.1 (reduction induced by a HDVF Gonzalez-Lorenzo 2016)**.** *A HDVF (P,S,C) induces a reduction (f,g,h) from (K,∂) to* (Span C,d) *with:*

$$g := \begin{array}{c} \\ C \\ P \\ S \\ C \end{array}\!\!\begin{pmatrix} 0 \\ G \\ id \end{pmatrix} \qquad f := \begin{array}{ccc} P & S & C \end{array}\!\!\begin{pmatrix} F & 0 & id \end{pmatrix}\ C \qquad h := \begin{array}{ccc} P & S & C \end{array}\!\!\begin{pmatrix} 0 & 0 & 0 \\ H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{c} P \\ S \\ C \end{array} \qquad d := \begin{array}{c} C \end{array}\!\!\begin{pmatrix} D \end{pmatrix}\ C$$

where $H = (\partial_S^P)^{-1}$, $F = -\partial_S^C \cdot H$, $G = -H \cdot \partial_C^P$ *and* $D = \partial_C^C - \partial_S^C \cdot H \cdot \partial_C^P$. *Similarly we can write:*

$$\begin{aligned} h &= \mathtt{i}_S \circ (\mathtt{j}_P \circ \partial \circ \mathtt{i}_S)^{-1} \circ \mathtt{j}_P \\ f &= \mathtt{j}_C \circ (id - \partial h) \\ g &= (id - h\partial) \circ \mathtt{i}_C \\ d &= \mathtt{j}_C \circ (\partial + \partial h \partial) \circ \mathtt{i}_C = f \circ \partial \circ \mathtt{i}_C = \mathtt{j}_C \circ \partial \circ g \end{aligned}$$

An example of a simple HDVF and its associated reduction. Let us state Proposition 1.6.1 again, with color code. A HDVF (P,S,C) induces a reduction (f,g,h) from (K,∂) to $(SpanC,d)$ with:

$$g := \begin{matrix} C \\ \begin{pmatrix} 0 \\ G \\ id \end{pmatrix} \end{matrix} \begin{matrix} P \\ S \\ C \end{matrix} \qquad f := \begin{matrix} P & S & C \\ \begin{pmatrix} F & 0 & id \end{pmatrix} \end{matrix} \; C \qquad h := \begin{matrix} P & S & C \\ \begin{pmatrix} 0 & 0 & 0 \\ H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \begin{matrix} P \\ S \\ C \end{matrix} \qquad d := \begin{matrix} C \\ \begin{pmatrix} D \end{pmatrix} \end{matrix} \; C$$

where $H = (\partial_S^P)^{-1}$, $F = -\partial_S^C \cdot H$, $G = -H \cdot \partial_C^P$ and $D = \partial_C^C - \partial_S^C \cdot H \cdot \partial_C^P$

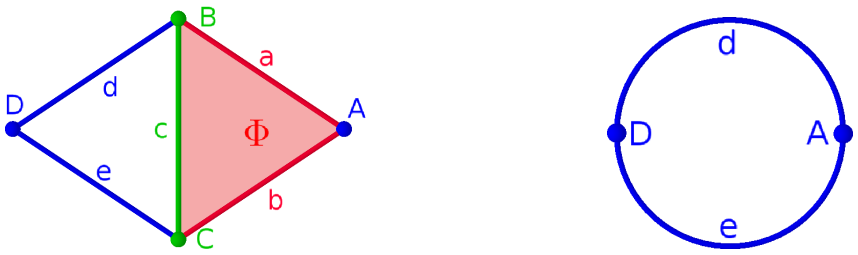


Figure 1.17 – A HDVF (P,S,C) on a complex (K,∂) (left) and a representation of its reduced complex (Span C,d) (right).

In the example of Figure 1.17, Proposition 1.6.1 implies the following matrices:

$$\partial_1 = \begin{matrix} a & b & c & d & e \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \qquad \partial_2 = \begin{matrix} \Phi \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} \qquad H_1 = \begin{matrix} c \\ \begin{pmatrix} 1 \end{pmatrix} \end{matrix} \; \Phi \qquad H_0 = \begin{matrix} B & C \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix} \begin{matrix} a \\ b \end{matrix}$$

$$d_1 = D_1 = \partial^{C_0}_{C_1} - \partial^{C_0}_{S_1} \cdot H_0 \cdot \partial^{P_0}_{C_1} \qquad d_2 = 0 \qquad d_0 = \begin{matrix} A & D \end{matrix} \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{matrix} d & e \end{matrix} \begin{matrix} \\ A \\ D \end{matrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

As reductions preserve homology, the homology of (K,∂) is the same as the homology of $(\operatorname{Span} C, d)$. In particular:

$$H_1(K) \approx \frac{\ker d_1}{\operatorname{im} d_2} = \frac{\operatorname{Span}\{d+e\}}{\{0\}} \approx \mathbb{Z}/2\mathbb{Z}$$

Perfect HDVFs.

Definition 1.6.2 (perfect HDVF)**.** A HDVF X is said to be *perfect* if its associated reduction is perfect, i.e. if $d = 0$.

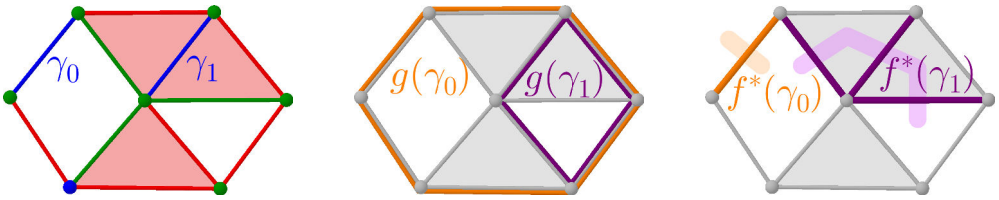


Figure 1.18 – A perfect HDVF (P,S,C) with $C_1 = \{\gamma_0,\gamma_1\}$ (left), its associated 1-homology basis $\{g(\gamma_0), g(\gamma_1)\}$ (middle) and 1-cohomology basis $\{f^*(\gamma_0), f^*(\gamma_1)\}$ (right).

In this case, a perfect HDVF provides detailed homological information. First, $\operatorname{Span} C_q \approx H_q(K)$, so the number of critical q-cells coincides with the number of q-holes. Second, all properties of perfect reductions stated in Section 1.4.2 apply.

In particular, $\{g_q(\gamma)\}_{\gamma\in C_q}$ is a q-homology basis, which we call the *q-homology basis induced by X*, see Figure 1.18. Dually, $\{f^*_q(\gamma)\}_{\gamma\in C_q}$ is the *q-cohomology basis induced by X*. The explicit isomorphisms $\operatorname{Span} C_q \approx H_q(K)$ and $\operatorname{Span} C_q \approx H^q(K)$ are given by the following lemma.

Lemma 1.6.2. *Let (P,S,C) be a perfect HDVF with reduction (f,g,h). Then, $[\cdot]_q \circ g_q$ is an isomorphism from $\operatorname{Span} C_q$ to $H_q(K)$, and its inverse is $f_q \circ s_q$.*

Dually, $[\cdot]^q \circ f^*_q$ is an isomorphism from $\operatorname{Span} C_q$ to $H^q(K)$, and its inverse is $g^*_q \circ s^q$.

Proof. As the reduction is perfect, the map g induces a q-homology basis $\{g_q(\gamma)\}_{\gamma \in C_q}$, which directly implies that $[\cdot]_q \circ g_q$ is an isomorphism from $\operatorname{Span} C_q$ to $H_q(K)$. Then, we have the following equalities (we omit the subscript q):

$$\begin{aligned}
[\cdot]g \circ fs &= [\cdot](id - \partial h - h\partial)s && \text{by Definition 1.4.2} \\
&= [\cdot]s - [\cdot]\partial hs - [\cdot]h\partial s \\
&= id && \text{as } [\cdot]\circ s = id,\ [\cdot]\circ\partial = 0 \text{ and } \partial\circ s = 0
\end{aligned}$$

Thus, fs is the inverse of $[\cdot]g$. □

A particularly important property of perfect HDVFs, which we use extensively in our work, is stated in Lemma 1.6.3. To formulate it we first define two maps which we call the *canonical cycle* and *cocycle* maps, that generalizes g and f^* for non-critical cells. Given a HDVF (P,S,C) with reduction (f,g,h), they are respectively defined as follows:

$$z_q = id - h_{q-1}\partial_q \qquad z^q = id - h_q^*\partial_{q+1}^*$$

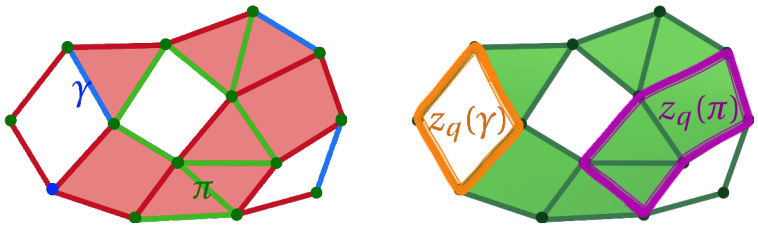


Figure 1.19 – A perfect HDVF (P,S,C) with two highlighted cells $\gamma \in C$ and $\pi \in P$ (left). The canonical cycles $z_q(\gamma)$ and $z_q(\pi)$ (right). $z_q(\gamma)$ (resp. $z_q(\pi)$) is the only cycle composed of secondary cells and γ (resp. π). As γ is critical, its canonical cycle is equal to its corresponding generator $g(\gamma)$. For a primary cell, the canonical cycle may or may not be a boundary. In the case of π, it is a boundary. For a secondary cell, the canonical cycle is 0: no cycle can be composed exclusively of secondary cells.

Lemma 1.6.3. *In a perfect HDVF (P,S,C):*

$$\forall x \in K_q \qquad (x + \operatorname{Span} S) \cap \ker\partial = \{z_q(x)\} \qquad (x + \operatorname{Span} P) \cap \ker\partial^* = \{z^q(x)\}$$

In other words, $z_q(x)$ (resp. $z^q(x)$) is the one and only cycle (resp. cocycle) composed of x and secondary (resp. primary) cells (see Figure 1.19 for an illustration).
In particular, if $p \in \operatorname{Span} P$, $s \in \operatorname{Span} S$ and $c \in \operatorname{Span} C_q$ we have

$$\begin{aligned}
(s + \operatorname{Span} S) \cap \ker\partial &= \{0\} = \{z_q(s)\} & (p + \operatorname{Span} P) \cap \ker\partial^* &= \{0\} = \{z^q(p)\} \\
(c + \operatorname{Span} S) \cap \ker\partial &= \{g_q(c)\} & (c + \operatorname{Span} P) \cap \ker\partial^* &= \{f_q^*(c)\}
\end{aligned}$$

Proof. We only prove the first equation as the second equation is the same result stated on the dual complex (see Definition 1.3.12 and Proposition 1.6.5 for more details on this duality).

First, $z_q(x) \in x + \operatorname{Span} S$ because $z_q(x) = x - h\partial(x)$ and $\operatorname{im} h \subseteq \operatorname{Span} S$. Moreover, as $gf = id - \partial h - h\partial$, we obtain $z_q(x) = gf(x) + \partial h(x)$. Thus, $z_q(x) \in \ker \partial$, because $gf(x)$ and $\partial h(x)$ are cycles (as the HDVF is perfect).

Second, suppose $x + s \in (x + \operatorname{Span} S) \cap \ker \partial$. This implies that $y := z_q(x) - (x + s)$ is a cycle in $\operatorname{Span} S$. With the reduction properties we have $y = gf(y) + \partial h(y) + h\partial(y) = 0$ (because $h \circ \mathtt{i}_S = 0$, $\partial(y) = 0$ and $f \circ \mathtt{i}_S = 0$, see Definition 1.4.2). Hence, $x + s = z_q(x)$. □

This lemma captures several key properties of perfect HDVFs, and is a central tool in a lot of the proofs in the following chapters. Moreover, it establishes a direct links between HDVFs and the notion of tri-partitions introduced later, see Section 2.2. A more detailed discussion about this lemma is given in Section 2.2.3.

DGVF vs HDVF. Contrary to DGVFs, HDVFs satisfy the following central property.

Proposition 1.6.4 (Proposition 6.8 in GONZALEZ-LORENZO, BAC, MARI, et al. 2017)**.** *When homology is computed over a field, every combinatorial chain complex admits a perfect HDVF.*

Moreover, a perfect HDVF can be computed greedily. We start with a partition consisting of only critical cells, and we greedily pair critical cells, turning them into primary and secondary cells, while maintaining the invertibility of ∂_S^P. This procedure is called a *completion* and is described in Section 1.6.3.

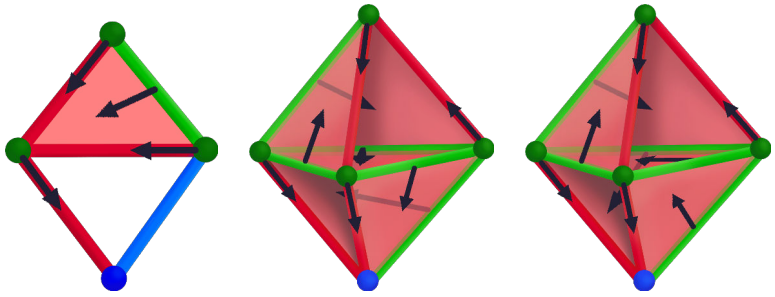
It was proved in GONZALEZ-LORENZO, BAC, MARI, et al. 2017 that every DGVF induces a HDVF. Precisely, primary and secondary cells correspond to sources and targets of reverted arrows, respectively. Critical cells thus correspond to cells that do not belong to any reverted arrows. A stronger result is that every iterated DGVF is a HDVF. Proposition 1.6.4 implies that HDVFs strictly generalize DGVFs. A simple example of a HDVF that does not come from a DGVF is shown in Figure 1.20.

Note that it is still possible to represent a HDVF using arrows from primary to secondary cells, as for the reversed arrows in DGVFs, but several choices of arrows may be possible, see Figure 1.20. In this manuscript we represent HDVFs only through their partition, with green, red and blue cells.

Duality and HDVFs. The duality between homology and cohomology also applies to HDVF:

Proposition 1.6.5. *If (P,S,C) is a HDVF for (K,∂) with reduction (f,g,h), then (S,P,C) is a HDVF for the dual complex $(K,\partial)^*$ with reduction (g^*,f^*,h^*).*

$$\begin{array}{ccccccccc}
\cdots & \underset{h_q^*}{\overset{\partial_{q+1}^*}{\leftrightarrows}} & K_q & \underset{h_{q-1}^*}{\overset{\partial_q^*}{\leftrightarrows}} & K_{q-1} & \underset{h_{q-2}^*}{\overset{\partial_{q-1}^*}{\leftrightarrows}} & \cdots \\
 & & f_q^* \uparrow \downarrow g_q^* & & f_{q-1}^* \uparrow \downarrow g_{q-1}^* & & \\
\cdots & \xleftarrow{d_{q+1}^*} & \operatorname{Span} C_q & \xleftarrow{d_q^*} & \operatorname{Span} C_{q-1} & \xleftarrow{d_{q-1}^*} & \cdots
\end{array}$$



(a) (b) (c)

Figure 1.20 – HDVFs represented using arrows from primary to secondary cells, mimicking the reversed arrows of DGVFs. The arrows of (a) actually form a (perfect) DGVF, whereas the arrows of (b) and (c) do not form valid DGVFs, as their induced Morse graphs contain cycles. In fact, the HDVF displayed in (b) and (c) is perfect and does not come from any DGVF. (b) and (c) also illustrate two different arrow choices for the same HDVF.

See Figure 1.21 for an illustration.

In particular, (P,S,C) is perfect for (K,∂) if and only if (S,P,C) is perfect for $(K,\partial)^*$. Moreover, the homology generators induced by (P,S,C) correspond to the cohomology generators induced by (S,P,C), and vice versa.

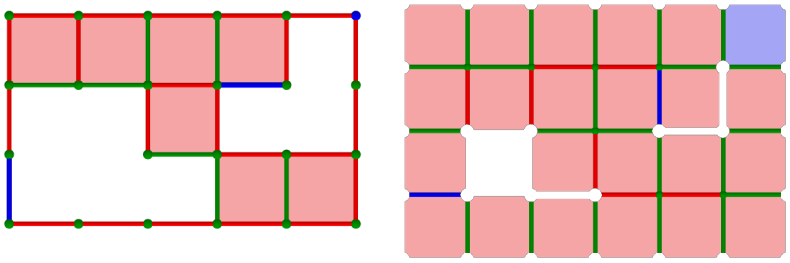


Figure 1.21 – A perfect HDVF (P,S,C) for the complex (K,∂) (on the left), and its dual HDVF (S,P,C) for $(K,\partial)^*$ (on the right).

We conclude this subsection with another property of perfect HDVFs, which illustrates once again how HDVFs interleave the notions of cells and chains.

Lemma 1.6.6 (structure of (co)boundaries)**.** *A perfect HDVF (P,S,C) for (K,∂) satisfies the following dual properties:*

1. *A boundary has at least one primary cell:* $\operatorname{im}\partial \cap \operatorname{Span}(S \sqcup C) = \{0\}$.
2. *A coboundary has at least one secondary cell:* $\operatorname{im}\partial^* \cap \operatorname{Span}(P \sqcup C) = \{0\}$.

Proof. Let $x \in \operatorname{im}\partial \cap \operatorname{Span}(S \sqcup C)$: there exist y, $s \in \operatorname{Span} S$ and $c \in \operatorname{Span} C$ such that $x = \partial y = s + c$. We have:

$$\begin{aligned} f\partial(y) &= f(s) + f(c) && \\ df(y) &= f(c) && \text{because } f\partial = df \text{ and } f \circ \mathtt{i}_S = 0 \\ 0 &= f(c) && \text{because } d = 0 \text{ as the HDVF is perfect} \\ 0 &= c && \text{because } f \circ \mathtt{i}_C = id_C \text{ by Proposition 1.6.1} \end{aligned}$$

As a result: $x = s \in \operatorname{Span} S$ and $x \in \operatorname{im}\partial \subseteq \ker\partial$. By Lemma 1.6.3 we obtain $x = 0$.

Doing the same proof with the dual objects (replacing f, ∂, d, and P by g^*, ∂^*, d^* and S) proves the second equality. □

About the name. The name of HDVFs has been debated since their introduction. The name *homological discrete vector field* comes from its original definition as an extension of *discrete gradient vector field*. However, unlike DGVFs, arrows are not fundamental in HDVFs, only the status of cells matters. For this reason, the term can be considered misleading and long. Thus, alternative names have been proposed.

- The term *homological coloring* highlights the labeling aspect. However, the term *coloring* usually refers to labelings where colors play symmetric interchangeable roles, which is not the case here.
- The term *cellular reduction* highlights that HDVFs come with a reduction defined from a notion of cells. However, the term *reduction* is already widely used for many different purposes, which may lead to confusion.
- The term *tri-partition*, used in H. Edelsbrunner and Ölsböck 2020 for a related concept, emphasizes the partition of cells into three classes. However, it lacks explicit connection with homology.

Finally, because no alternative names seemed clearly better than the others, the term *homological discrete vector field* remained in our daily usage and stayed as the standard one.

1.6.2. Elementary swapping operations

In Gonzalez-Lorenzo, Bac, Mari, et al. 2017, the authors define several elementary operations to modify locally a HDVF. Notably, they define three swapping operations $\mathtt{M}$, $\mathtt{W}$ and $\diamondsuit$ which operate local modifications of HDVFs by swapping the status of two cells. In general, swapping the status of two cells transforms a HDVF into a simple partition (P, S, C), we say that an operation is *valid* for a HDVF X if the resulting partition is also a HDVF (see Figure 1.22 for some examples).

Definition 1.6.3 (HDVF operations Gonzalez-Lorenzo 2016)**.** Given a HDVF (P, S, C) with reduction (f, g, h):

1. The $\mathtt{W}$ operation swaps a secondary cell with a critical cell. If $\sigma \in S$ and $\gamma \in C$:

$$\mathtt{W}_\sigma^\gamma\,(P,S,C) = \big(P\,,\ S\cup\{\gamma\}\backslash\{\sigma\},\ C\cup\{\sigma\}\backslash\{\gamma\}\big)$$

$\mathtt{W}_\sigma^\gamma$ is valid for X if and only if $\langle g(\gamma),\sigma\rangle$ is invertible[5].
Equivalently, $\mathtt{W}_\sigma^\gamma$ is valid for X if and only if $\langle z_q(\gamma),\sigma\rangle$ is invertible.

2. The $\mathtt{M}$ operation swaps a primary cell with a critical cell. If $\pi\in P$ and $\gamma\in C$:

$$\mathtt{M}_\pi^\gamma\,(P,S,C) = \big(P\cup\{\gamma\}\backslash\{\pi\},\ S,\ C\cup\{\pi\}\backslash\{\gamma\}\big)$$

$\mathtt{M}_\pi^\gamma$ is valid for X if and only if $\langle f(\pi),\gamma\rangle$ is invertible.
Equivalently, $\mathtt{M}_\pi^\gamma$ is valid for X if and only if $\langle z^q(\gamma),\pi\rangle$ is invertible.

3. The $\diamondsuit$ operation swaps a primary cell with a secondary cell. If $\pi\in P$ and $\sigma\in S$:

$$\diamondsuit_\pi^\sigma\,(P,S,C) = (P\cup\{\sigma\}\backslash\{\pi\},\ S\cup\{\pi\}\backslash\{\sigma\},\ C)$$

$\diamondsuit_\pi^\sigma$ is valid for X if and only if $\langle\partial h(\pi),\sigma\rangle$ and $\langle h\partial(\pi),\sigma\rangle$ are invertible.
Equivalently, $\diamondsuit_\pi^\sigma$ is valid for X if and only if $\langle z_q(\pi),\sigma\rangle$ and $\langle z^q(\sigma),\pi\rangle$ are invertible.

Note that these valid operations transform a perfect HDVF into another perfect HDVF because they do not change the number of critical cells. In what follows, we will only consider valid operations.

Gonzalez-Lorenzo 2016 also proved that the three operations $\mathtt{M}$, $\mathtt{W}$ and $\diamondsuit$ can be generalized to sets of cells. Precisely, given a HDVF $X=(P,S,C)$ with induced reduction (f,g,h), and $\Pi\subseteq P$, $\Sigma\subseteq S$ and $\Gamma\subseteq C$, we can define the generalized operations $\mathtt{W}_\Sigma^\Gamma$, $\mathtt{M}_\Pi^\Gamma$ and $\diamondsuit_\Pi^\Sigma$ which swap the status of set of cells as for Definition 1.6.3. The validity of these generalized operations is similar to the validity conditions in Definition 1.6.3, except that scalar products are replaced by matrix multiplications, and being different from zero is replaced by being invertible. For example, $\mathtt{W}_\Sigma^\Gamma$ is a valid operation for X if and only if $\mathtt{j}_\Sigma\circ g\circ\mathtt{i}_\Gamma$ is invertible.

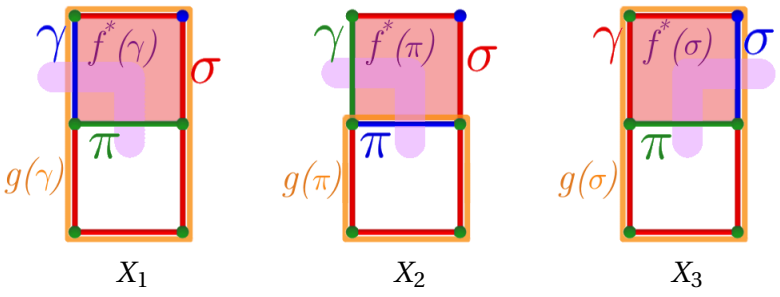


Figure 1.22 – Three perfect HDVFs on the same complex such that $X_2=\mathtt{M}_\pi^\gamma X_1$ and $X_3=\mathtt{W}_\sigma^\gamma X_1$. As stated in Proposition 1.6.7, the $\mathtt{M}$ operation preserves the cohomology generator (represented with purple dual edges): $f^*(\gamma)$ in X_1 equals $f^*(\pi)$ in X_2. Similarly, the $\mathtt{W}$ operation preserves the homology generator (represented in orange): $g(\gamma)$ in X_1 equals $g(\sigma)$ in X_3.

[5] In a field, being invertible is equivalent to being non-zero. Thus, this condition is equivalent to $\sigma\in\overline{g(\gamma)}$.

Advanced properties of elementary swapping operations. We present some more advanced properties of $\mathtt{W}$, $\mathtt{M}$ and $\diamondsuit$. The two first properties are about how (co)homology generators, as well as canonical (co)cycles, are modified after an operation. Interestingly, $\mathtt{W}$ (respectively $\mathtt{M}$) operations preserve the homology (respectively cohomology) generator while changing the cohomology (respectively homology) generator associated to the involved critical cell:

Proposition 1.6.7 (Proposition 3.20 in GONZALEZ-LORENZO 2016)**.** *Given a perfect HDVF $X = (P, S, C)$ with induced reduction (f, g, h) and $\pi \in P$, $\sigma \in S$, $\gamma \in C$ we have:*

1. *the homology generator associated to γ in X is the same as the homology generator associated to σ in $\mathtt{W}_\sigma^\gamma X$;*
2. *the cohomology generator associated to γ in X is the same as the cohomology generator associated to π in $\mathtt{M}_\pi^\gamma X$.*

This property is illustrated in Figure 1.22

Hence, it is possible to modify the cohomology generators while preserving the homology generators using $\mathtt{M}$ operations and vice versa with $\mathtt{W}$ operations. A complement of Proposition 1.6.7 is the following lemma, which details the effect of the operations $\mathtt{M}$, $\mathtt{W}$ and $\diamondsuit$ on the canonical cycle and cocycle maps z_q and z^q.

Lemma 1.6.8 (canonical (co)cycles after operations)**.** *Let X and X' be two perfect HDVF with canonical maps (z_q, z^q) and (z'_q, z'^q), respectively Then we have for every $x \in K$:*

1. *If $X' = \mathtt{W}_\sigma^\gamma X$, denote $\xi := \langle z_q(\gamma), \sigma\rangle^{-1}$, we have:*

$$z'_q(x) = z_q(x) - \xi\langle z_q(x) - x, \sigma\rangle z_q(\gamma) \qquad z'^q(x) = z^q(x)$$

2. *If $X' = \mathtt{M}_\pi^\gamma X$, denote $\xi := \langle z^q(\gamma), \pi\rangle^{-1}$, we have:*

$$z'_q(x) = z_q(x) \qquad z'^q(x) = z^q(x) - \xi\langle z^q(x) - x, \pi\rangle z^q(\gamma)$$

3. *If $X' = \diamondsuit_\pi^\sigma X$, denote $\xi_q = \langle z_q(\pi), \sigma\rangle^{-1}$ and $\xi^q := \langle z^q(\sigma), \pi\rangle^{-1}$, we have:*

$$z'_q(x) = z_q(x) - \xi_q\langle z_q(x) - x, \sigma\rangle z_q(\pi) \qquad z'^q(x) = z^q(x) - \xi^q\langle z^q(x) - x, \pi\rangle z^q(\sigma)$$

Proof.
1. $X' = \mathtt{W}_\sigma^\gamma X$.

- Consider first the canonical cycle map z'_q. Let $y := z_q(x) - \xi\langle z_q(x) - x, \sigma\rangle z_q(\gamma)$. Our goal is to use Lemma 1.6.3 to prove $y = z'_q(x)$. y is a cycle, as a sum of the two cycles $z_q(x)$ and $z_q(\gamma)$. Thus, it remains to prove that $y - x \in \operatorname{Span} S'$.

  Denote $S^- = S\backslash\{\sigma\} = S'\backslash\{\gamma\}$. Using Lemma 1.6.3, we decompose $z_q(x) \in x +$

Span $S^- +$ Span $\{\sigma\}$ and $z_q(\gamma) \in \gamma +$ Span $S^- +$ Span $\{\sigma\}$ as follows:

$$z_q(x) = x + s_x + \langle z_q(x) - x, \sigma\rangle\sigma \qquad\qquad z_q(\gamma) = \gamma + s_\gamma + \xi^{-1}\sigma$$

with s_x and s_γ in Span S^-.

To ease notation, denote $\lambda := \langle z_q(x) - x, \sigma\rangle$. We get the following equalities:

$$\begin{aligned} y - x &= z_q(x) - \xi\lambda z_q(\gamma) \; - \; x \\ &= s_x + \lambda\sigma \; - \; \xi\lambda(\gamma + s_\gamma + \xi^{-1}\sigma) \\ &= s_x - \xi\lambda(\gamma + s_\gamma) + \lambda(1 - \xi\xi^{-1})\sigma \\ &= s_x - \xi\lambda(\gamma + s_\gamma) \end{aligned}$$

Hence, as $\gamma \in S'$ and both s_x and s_γ are in Span $S^- \subseteq$ Span S', we get $y - x \in$ Span S'. Using Lemma 1.6.3 we get $y = z'_q(x)$.

- Consider now the canonical cocycle map z'^q. We have $P = P'$, hence, Lemma 1.6.3 directly gives $z^q = z'^q$.

2. $X' = \mathtt{M}_\pi^\gamma X$. The proof of this case is the dual of the proof for $X' = \mathtt{W}_\sigma^\gamma X$. Precisely, the roles of z_q and z^q are inverted, as well as the roles of P and S.
3. $X' = \diamondsuit_\pi^\sigma X$. For the canonical cycle map z'_q, the proof is the same as for $X' = \mathtt{W}_\sigma^\gamma X$, but replacing γ with π. For the canonical cocycle map z'^q, the proof is the same as for $X' = \mathtt{M}_\pi^\gamma X$, but replacing γ with σ. □

To conclude this subsection, we present a proposition that shows how we can sometimes combine two consecutive operations.

Lemma 1.6.9 (ternary operations)**.** *Given $X = (P, S, C)$ a perfect HDVF with induced (co)cycle projections (z_q, z^q):*

1. $\diamondsuit_\gamma^\sigma \mathtt{M}_\pi^\gamma$ *is valid for X if $\langle z_q(\gamma), \sigma\rangle$, $\langle z^q(\gamma), \pi\rangle$ and $\langle z^q(\sigma), \pi\rangle$ are invertible.*
2. $\diamondsuit_\pi^\gamma \mathtt{W}_\sigma^\gamma$ *is valid for X if $\langle z_q(\gamma), \sigma\rangle$, $\langle z^q(\gamma), \pi\rangle$ and $\langle z_q(\pi), \sigma\rangle$ are invertible.*

Proof. We only prove the first assumption, as the second uses dual arguments. Firstly, as $\langle z^q(\gamma), \pi\rangle$ is invertible, $\mathtt{M}_\pi^\gamma$ is a valid operation for X. Denote $X' = \mathtt{M}_\pi^\gamma X$ and its associated canonical cycle map z'_q. By Lemma 1.6.8, we have $z'_q(x) = z_q(x)$, and $z'^q(x) = z^q(x) - \xi\langle z^q(x) - x, \pi\rangle z^q(\gamma)$, with $\xi = \langle z^q(\gamma), \pi\rangle^{-1}$.

Note that γ is a primary cell in X'. We obtain the following equalities:

$$\begin{aligned} &\langle z'_q(\gamma), \sigma\rangle = \langle z_q(\gamma), \sigma\rangle \quad \text{is invertible} && \text{by assumption} \\ &\langle z'^q(\sigma), \gamma\rangle = \big\langle z^q(\sigma) - \xi\langle z^q(\sigma) - \sigma, \pi\rangle z^q(\gamma),\ \gamma\big\rangle \\ &\qquad = \langle z^q(\sigma), \gamma\rangle - \xi\langle z^q(\sigma), \pi\rangle\langle z^q(\gamma), \gamma\rangle && \text{because } \langle \sigma, \pi\rangle = 0 \\ &\qquad = -\xi\langle z^q(\sigma), \pi\rangle\langle z^q(\gamma), \gamma\rangle && \text{as } z^q(\sigma) \in \sigma + \text{Span}\, P \text{ but } \gamma \notin \text{Span}\, P \\ &\qquad = -\xi\langle z^q(\sigma), \pi\rangle && \text{as } \langle z^q(\gamma), \gamma\rangle = 1 \end{aligned}$$

Thus, $\langle z'^q(\sigma), \gamma\rangle$ is invertible as both ξ and $\langle z^q(\sigma), \pi\rangle$ are invertible, by assumption. As a result, by Definition 1.6.3, $\Diamond_\gamma^\sigma$ is a valid operation for X'. Hence, $\Diamond_\gamma^\sigma \mathtt{M}_\pi^\gamma$ is valid for X. □

1.6.3. Completion of HDVF

A core result of GONZALEZ-LORENZO 2016 is the following: in a non perfect HDVF, with coefficients over a field, it is always possible to pair two critical cells (i.e. transform two critical cells to one primary and one secondary cell) in order to obtain HDVF with fewer critical cells. This pairing operation is denoted by $\mathtt{A}$.

Definition 1.6.4 (pairing operation)**.** Given a HDVF (P,S,C) with reduction (f,g,h) from (K,∂) to $(\operatorname{Span} C, d)$, the operation $\mathtt{A}$ transforms two critical cells into a primary and a secondary. If $\pi \in C$ and $\sigma \in C$:

$$\mathtt{A}_\pi^\sigma\ (P,S,C) = (P \cup \{\pi\},\ S \cup \{\sigma\},\ C \backslash \{\pi,\sigma\})$$

$\mathtt{A}_\pi^\sigma$ is valid for X if and only if $\langle d(\sigma), \pi\rangle$ is invertible.

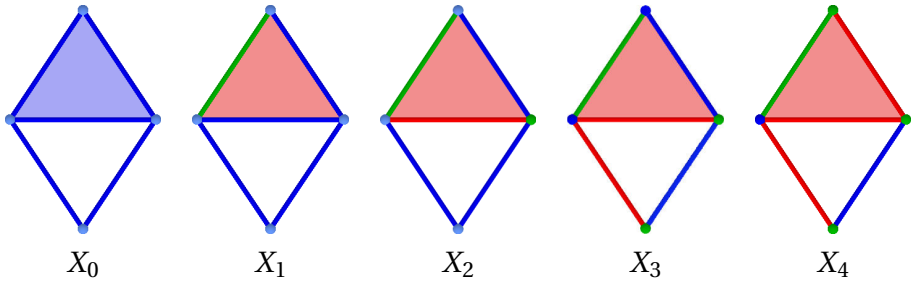


Figure 1.23 – Five HDVFs on the same complex. X_0 is the "empty" HDVF, where every cell is critical. Every other HDVF X_i is obtained as a completion of the previous HDVF: $X_i = \mathtt{A}_\pi^\sigma X_{i-1}$ for some cells π and σ. X_4 is a perfect HDVF.

GONZALEZ-LORENZO, BAC, MARI, et al. 2017 also detailed how the reduction changes after performing such a pairing.

Therefore, for homology over a field, it is always possible to greedily perform $\mathtt{A}$ operations until obtaining a perfect HDVF. This process is called *the completion* of a HDVF, see Figure 1.23.

Proposition 1.6.10 (completion of a HDVF GONZALEZ-LORENZO 2016)**.** *Given $X = (P,S,C)$ a HDVF for (K,∂), there exists $X' = (P',S',C')$ a perfect HDVF for (K,∂) such that $P \subseteq P'$ and $S \subseteq S'$.*

Every HDVF X' such that $P \subseteq P'$ and $S \subseteq S'$ is said to be a *completion* of X. In this case, we write $X \prec X'$.

Given two complexes $(K,\partial) \subseteq (K',\partial')$, a HDVF $X = (P,S,C)$ for the subcomplex (K,∂) can also be considered as a HDVF $(P,S,C \cup (\overline{K'} \backslash \overline{K}))$ for the bigger complex (K',∂'). Indeed, the invertibility assumption is still satisfied in this new HDVF. Using this observation, we get the following corollary.

Corollary 1.6.11. *Given two complexes $(K,\partial) \subseteq (K',\partial')$ and a HDVF X for (K,∂), there exists a perfect HDVF X' for (K',∂') such that $X \prec X'$.*

Advanced completion properties. We conclude this section with three additional lemmas about this completion process.

First, the $\mathtt{A}$ operation can also be generalized for sets of cells. Given Π, Σ disjoint subsets of critical cells of a HDVF (P,S,C):

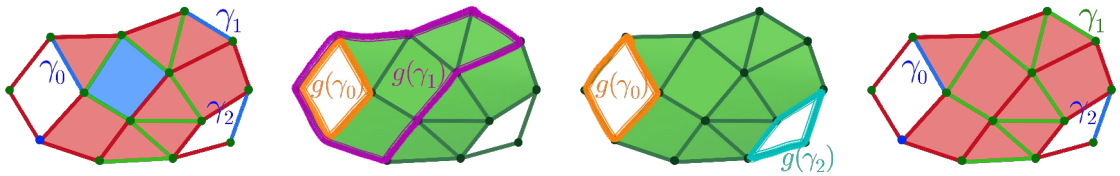
$$\mathtt{A}^{\Sigma}_{\Pi}(P,S,C) = (P \sqcup \Pi,\ S \sqcup \Sigma,\ C \setminus (\Pi \sqcup \Sigma))$$

Lemma 1.6.12. *$\mathtt{A}^{\Sigma}_{\Pi}$ is valid for X if and only if $\mathtt{j}_{\Pi} \circ d \circ \mathtt{i}_{\Sigma}$ is invertible, where d is the reduced boundary of X.*

Proof. The proof is the same as for Proposition 3.11 and 3.19 in GONZALEZ-LORENZO 2016, using block matrices. □

Second, during the completion process, it is possible to preserve some critical cells. Precisely, every critical cell γ such that $g(\gamma)$ is a cycle that is not a boundary can be preserved. This is stated formally in the following technical lemma.

Lemma 1.6.13 (critical preservation during completion)**.** *Let $X = (P,S,C)$ be a HDVF for (K,∂) with reduction (f,g,h). Suppose that there is $C^* \subseteq C_q$ such that for all $\gamma \in C^*$, $g(\gamma)$ is a cycle, and such that $([g_q(\gamma)]^K)_{\gamma \in C^*}$ is linearly independent in $H_q(K)$.*
Then, there exists $X' = (P',S',C')$ a perfect HDVF for (K,∂) such that $X \prec X'$ and $C^* \subseteq C'$.



X $\quad C^* = \{\gamma_0, \gamma_1\}$ $\quad C^* = \{\gamma_0, \gamma_2\}$ $\quad X'$

Figure 1.24 – Illustration of Lemma 1.6.13. The first image is a HDVF $X = (P,S,C)$, with $C_1 = \{\gamma_0, \gamma_1, \gamma_2\}$. The second image represents a set $C^* = \{\gamma_0, \gamma_1\}$ that does not satisfy the assumptions of Lemma 1.6.13. Indeed, although $g(\gamma_0)$ and $g(\gamma_1)$ are cycles, $\{[g(\gamma_0)], [g(\gamma_1)]\}$ is not linearly independent (they represent the same hole). The third image shows a set $C^* = \{\gamma_0, \gamma_2\}$ that satisfies the assumptions of Lemma 1.6.13. The last image is the resulting perfect HDVF X' that preserves both γ_0 and γ_2 as critical cells, and such that $X \prec X'$.

Proof. This proof is long and technical. We decided to put it in appendix, to avoid overwhelming the reader, see Proof of Lemma 1.6.13. Instead, we give an illustration of this lemma, see Figure 1.24. □

Considering two complexes $(K,\partial) \subseteq (K',\partial')$, and a perfect HDVF for (K,∂), Lemma 1.6.13 translates as follows: it is possible to complete a perfect HDVF for (K,∂) to a perfect HDVF for (K',∂') while preserving the critical cells associated with holes of (K,∂) that were not filled in (K',∂'). This is stated in the following corollary.

Corollary 1.6.14. *Let $(K,\partial) \subseteq (K',\partial')$ and let $X = (P,S,C)$ be a perfect HDVF for (K,∂) with reduction (f,g,h). Suppose that there is $C^* \subseteq C_q$ such that $([g_q(\gamma)]^{K'})_{\gamma\in C^*}$ is linearly independent in $H_q(K')$.*
Then, there exists $X' = (P',S',C')$ a perfect HDVF for (K',∂') such that $X \prec X'$ and $C^* \subseteq C'$.

Proof. The corollary is obtained by considering X as a HDVF for (K',∂') (not necessary perfect) and applying Lemma 1.6.13. The fact that $g(\gamma)$ is a cycle is ensured by the fact that X is perfect for (K,∂), which implies that $\partial g = 0$. □

The last lemma that we present here is a direct consequence of the canonical cycle Lemma 1.6.3. It states a condition under which a canonical (co)cycle is preserved during a completion.

Lemma 1.6.15 (canonical (co)cycle preservation during completion)**.** *Suppose $(K,\partial) \subseteq (K',\partial')$. Let $X = (P,S,C)$ be a HDVF (non necessarily perfect) for (K,∂) and $X' = (P',S',C')$ be perfect HDVFs for (K',∂') such that and $X \prec X'$.*
Then, $\forall x \in K_q$ such that $z_q(x) \in \ker\partial'$, we have $z_q(x) = z'_q(x)$.
Dually, $\forall x \in K_q$ such that $z^q(x) \in \ker\partial'^*$, we have $z^q(x) = z'^q(x)$.

Proof. Let $x \in K \subseteq K'$. By definition of z_q and because $\operatorname{im} h \in \operatorname{Span} S$, we have $z_q(x) \in x + \operatorname{Span} S$. As $X \prec X'$, we have $S \subseteq S'$ so $z_q \in x + \operatorname{Span} S'$.
Now, if $z_q(x) \in \ker\partial'$ we obtain $z_q(x) = z'_q(x)$ by Lemma 1.6.3 (as X' is perfect).
With dual arguments we get $z^q(x) \in \ker\partial'^* \cap (x + \operatorname{Span} P') = \{z'^q(x)\}$. □

1.6.4. Emulating standard computational homology methods

In this last subsection, we recall results from GONZALEZ-LORENZO 2016 showing how HDVFs encode standard computational homology approaches.

Smith normal form. It has been proved that the algorithm that computes perfect HDVF is equivalent to performing SNF reductions on the boundary matrix. Indeed, each A-operation corresponds to a Gaussian elimination step. Precisely, SNF of the q-boundary matrix ∂_q can be emulated by a HDVF.
Conversely, computing a perfect HDVF simultaneously computes an SNF of each boundary map ∂_q, with compatible bases in between dimensions. This is equivalent to computing a modified SNF, see PELTIER et al. 2006.

Discrete Morse theory. Since HDVFs strictly generalize DGVFs, they also encode the homology computations of discrete Morse theory.

Persistent homology. It is shown in GONZALEZ-LORENZO 2016 that the classical persistent homology algorithm based on SNF can be computed with a variation of the algorithm that computes perfect HDVFs. The idea is to perform A operations according to the input filtration. We refine this observation in Section 3.2, where we detail how HDVFs encode the standard abstract persistence algorithm of EDELSBRUNNER et al. 2002.

We further extend this list in Section 2.2, where we investigate the relation between HDVFs and *tri-partitions*, introduced in H. EDELSBRUNNER and ÖLSBÖCK 2020. These results highlight that HDVFs provide a framework for studying the homology computations themselves. To conclude, it is important to keep in mind that HDVFs are hybrid objects, in between algebra and combinatorics, interleaving the notions of cells and chains. This fact underlies the three following chapters.

2. Combinatorial Properties of HDVFs

Sommaire

2.1. Introduction 64
2.2. Tri-Partitions 65
2.2.1. Definition 65
2.2.2. Relation between Tri-partitions and perfect HDVFs 66
2.2.3. The canonical (co)cycle lemma 68
2.3. (Co)Homology Bases Computed by HDVF and Related Methods 69
2.3.1. Explicit homology bases 70
2.3.2. Characterization of the homology bases computed by HDVFs . 73
2.3.3. Explicit cohomology bases 78
2.3.4. Combining explicit homology and cohomology bases 80
2.3.5. Characterization of homology bases computed by other methods 83
2.4. Conclusion 84

2.1. Introduction

This chapter aims to refine the understanding of HDVFs. More specifically, we develop the combinatorial properties of perfect HDVFs.

First, Section 2.2 establishes a close relationship between perfect HDVFs and *tri-partitions*, a combinatorial structure introduced in H. EDELSBRUNNER and ÖLSBÖCK 2020. Tri-partitions are defined using the concept of *tree* and *co-tree*. This connection allows a more combinatorial interpretation of perfect HDVFs and their associated homology and cohomology bases. Conversely, it also refines the understanding of tri-partitions.

Next, Section 2.3 introduces the main contribution of this chapter: a characterization of the homology bases induced by perfect HDVFs. We introduce a class of cohomology and homology bases called *explicit* bases. We show that HDVFs can only compute explicit bases, and conversely, that every explicit basis can be computed by a HDVF. This characterization extends to other standard computational homology methods, which proves that common approaches compute only a small set of the possible homology bases.

These results highlight the significant role of combinatorics in computational homology approaches. Throughout this chapter, we consider combinatorial chain complexes, which we simply refer to as *complexes*. The results of this chapter were presented in the conference *Discrete Geometry and Mathematical Morphology 2025*, see GAZULL, GONZALEZ-LORENZO, et al. 2025a, and have been submitted for the corresponding special issue in *Journal of Mathematical Imaging and Vision*.

2.2. Tri-Partitions

2.2.1. Definition

H. EDELSBRUNNER and ÖLSBÖCK 2020 introduced the notion of *tri-partition*. A tri-partition is a combinatorial object providing homological information using the notions of tree and cotree, which generalize the notion of tree for graphs.

Definition 2.2.1 (trees and cotrees)**.** Let (K,∂) be a complex. A *q-tree* is a set of q-cells A that does not generate non-zero cycles, i.e. $\operatorname{Span} A \cap \ker\partial = \{0\}$. Similarly, a *$q$-cotree* is a set of q-cells A that does not generate non-zero cocycles, i.e. $\operatorname{Span} A \cap \ker\partial^* = \{0\}$. A q-tree A (resp. q-cotree) is maximal if there is no other q-tree (resp. q-cotree) A' such that $A \subset A'$.

Definition 2.2.2 (tri-partition)**.** Given a complex (K,∂), a *tri-partition* of dimension q is a triplet (A^q, A_q, E_q) forming a partition of the q-cells $\overline{K_q}$ such that A^q is a maximal q-cotree, A_q is a maximal q-tree and $|E_q| = \beta_q$. See Figure 2.1 for an example of a tri-partition.

Note that H. EDELSBRUNNER and ÖLSBÖCK 2020 introduced tri-partitions for polyhedral complexes. We directly present them for combinatorial chain complexes, as the definitions and proofs also work in this context. The authors then proved the following theorem about the existence of tri-partitions. This theorem was initially proved for homology over $\mathbb{Z}/2\mathbb{Z}$, but it holds for other fields.

Theorem 2.2.1 (Theorem 3.1 in H. EDELSBRUNNER and ÖLSBÖCK 2020)**.** *Let (K,∂) be a complex. Then, for all q, there exist a tri-partition of dimension q for (K,∂).*

In their article, the authors explain how to adapt Smith normal form algorithm to compute tri-partitions. In addition, they show that every tri-partition induces bases for homology, cohomology but also for cycles and cocycles. These bases are defined using the following maps, which are called *canonical (co)cycle maps*, and coincide with the canonical cycle maps defined for HDVF, as explained later in Section 2.2.3.

Proposition 2.2.2 (canonical (co)cycle)**.** *Let (A^q, A_q, E_q) be a tri-partition for a complex (K,∂). If $\tau \in A^q \sqcup E_q$, there is a unique q-cycle $\tilde{z}_q(\tau)$ in $\tau + \operatorname{Span} A_q$. By duality, if $\tau \in A_q \sqcup E_q$, there is a unique q-cocycle $\tilde{z}^q(\tau)$ in $\tau + \operatorname{Span} A^q$.*

$\tilde{z}_q(\tau)$ and $\tilde{z}^q(\tau)$ are called the canonical *cycle and cocycle of τ, respectively. They are extended linearly to two maps $\tilde{z}_q : K_q \to \ker\partial_q$ and $\tilde{z}^q : K_q \to \ker\partial^*_{q+1}$.*

Theorem 2.2.3 (Theorem 4.3 in H. EDELSBRUNNER and ÖLSBÖCK 2020)**.**

- $\{\tilde{z}_q(\tau)\}_{\tau \in A^q \sqcup E_q}$ *is a basis of* $\ker \partial_q$*;*
- $\{\tilde{z}_q(\tau)\}_{\tau \in E_q}$ *is a* q*-homology basis;*
- $\{\tilde{z}^q(\tau)\}_{\tau \in A_q \sqcup E_q}$ *is a basis of* $\ker \partial^*_{q+1}$*;*
- $\{\tilde{z}^q(\tau)\}_{\tau \in E_q}$ *is a* q*-cohomology basis.*

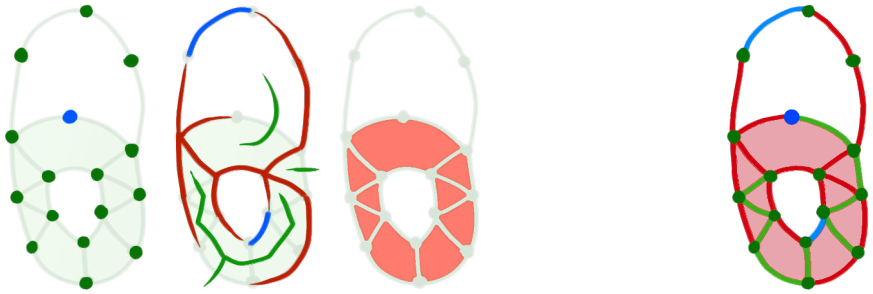


Figure 2.1 – Three tri-partitions (A^q, A_q, E_q) of dimension $q = 0$, 1 and 2 respectively. The fourth image is the perfect HDVF (P, S, C) obtained by "stacking" the three previous tri-partitions. Equivalently, the tri-partitions are the dimensional "layers" of the perfect HDVF, that is $(A^q, A_q, E_q) = (P_q, S_q, C_q)$. A^q, A_q and E_q are represented in green, red and blue, respectively, to match the color scheme of HDVFs, according to the result of Section 2.2.2. Note that the cotree A^1 is represented with transversal edges.

2.2.2. Relation between Tri-partitions and perfect HDVFs

The following propositions state that tri-partitions are closely related to perfect HDVFs. Precisely, they can be viewed as "dimensional layers" of perfect HDVFs. Conversely, a perfect HDVF can be viewed as a "stack" of tri-partitions.

Proposition 2.2.4. *If* (P, S, C) *is a perfect HDVF for* K*, then* (P_q, S_q, C_q) *is a* q*-tri-partition for every dimension* q*.*

Proof. By Lemma 1.6.3, we have $\operatorname{Span} S \cap \ker \partial = \{0\}$, so S_q is a q-tree. Dually, $\operatorname{Span} P \cap \ker \partial^* = \{0\}$, so P_q is a q-cotree. Moreover, $|C_q| = \beta_q$ because (P, S, C) is perfect.

It remains to prove the maximality of S_q and P_q. Let us consider S_q first: we prove that if we add a cell τ to S_q, it creates a cycle. Let τ be a q-cell that is not in S_q. As (P_q, S_q, C_q) is a partition of the q-cells, $\tau \in P_q \sqcup C_q$.

Consider $x := z_q(\tau) = \tau - h\partial(\tau)$ (by Definition 1.4.2). (P, S, C) is perfect so x is a cycle, by Lemma 1.6.3.

In addition, $\tau \notin S_q$, $\tau \neq 0$, and x is composed of cells in $S_q \cup \{\tau\}$ (by Lemma 1.6.3). Thus, x is a non-zero cycle of $S_q \sqcup \{\tau\}$, so S_q is maximal.

The proof of the maximality of P_q is similar: if $\tau \notin P_q$, then $z^q(\tau)$ is a non-zero cocycle with cells in $P_q \sqcup \{\tau\}$. □

We then show the converse: a "stack" of tri-partitions is a perfect HDVF:

Proposition 2.2.5. *If for every q, (A_q, A^q, E_q) is a q-tri-partition of K, then $(\bigcup A^q, \bigcup A_q, \bigcup E_q)$ is a perfect HDVF for K.*

Proof. We first prove that $(P,S,C) = (\bigcup A^q, \bigcup A_q, \bigcup E_q)$ is a HDVF. Thus, we need to show that $\partial^{A^q}_{A_{q+1}} := \mathtt{j}_{A^q} \circ \partial \circ \mathtt{i}_{A_{q+1}}$ is a square invertible matrix for every q.

To begin with, we prove the injectivity of $\partial^{A^q}_{A_{q+1}}$. Let $x \in \ker \partial^{A^q}_{A_{q+1}}$. We have $x \in \operatorname{Span} A_{q+1}$ and $\partial(x) \in \operatorname{Span} A_q \oplus \operatorname{Span} E_q$. Hence, there exists $t \in \operatorname{Span} A_q$ (a tree) and $e \in \operatorname{Span} E_q$ such that $\partial(x) = t + e$.

By Proposition 2.2.2, we have $\tilde{z}_q(e) - e \in \operatorname{Span} A_q$ and $\tilde{z}_q(e) \in \ker \partial$. This implies the following equalities:

$$\begin{aligned} \partial(x) - e &= t \\ \partial(x) - e - (\tilde{z}_q(e) - e) &= t - (\tilde{z}_q(e) - e) \\ \partial(x) - \tilde{z}_q(e) &= t - (\tilde{z}_q(e) - e) \end{aligned}$$

$\partial(x)$ and $\tilde{z}_q(e)$ are cycles, so $\partial(x) - \tilde{z}_q(e) \in \ker \partial$. In addition, t and $\tilde{z}_q(e) - e$ are in $\operatorname{Span} A_q$, so $\partial(x) - \tilde{z}_q(e) = t - (\tilde{z}_q(e) - e) \in \operatorname{Span} A_q$. Since $\operatorname{Span} A_q \cap \ker \partial = \{0\}$ (by definition of a q-tree), we obtain $\partial(x) - \tilde{z}_q(e) = 0$, so $\partial(x) = \tilde{z}_q(e)$.

This directly implies $[\tilde{z}_q(e)] = [\partial(x)] = 0$. By Theorem 2.2.3, $\{\tilde{z}_q(\epsilon)\}_{\epsilon \in E_q}$ is a homology basis, so $e = 0$ otherwise we would have $[\tilde{z}_q(e)] \neq 0$. Thus, we get $\partial(x) = t$, so $\partial(x) \in \operatorname{Span} A_q \cap \ker \partial = \{0\}$. Thus, $\partial(x) = 0$ and $x \in \ker \partial$. This implies that $x \in \operatorname{Span} A_{q+1} \cap \ker \partial = \{0\}$, so $x = 0$ and $\ker \partial^{A^q}_{A_{q+1}} = \{0\}$.

This directly implies the injectivity of $\partial^{A^q}_{A_{q+1}}$ and the fact that $|A_{q+1}| \leq |A^q|$.

By duality (doing the proof for $x \in \ker \partial^{*\,A^q}_{A_{q+1}}$), we obtain the injectivity of $\partial^{*\,A^q}_{A_{q+1}}$ and the fact that $|A_{q+1}| \geq |A^q|$. This implies that $\partial^{A^q}_{A_{q+1}}$ is square and injective, so it is invertible.

Hence, (P,S,C) is a HDVF. It is perfect because we have $|C_q| = |E_q| = \beta_q$. □

Propositions 2.2.4 and 2.2.5 underline the fact that perfect HDVFs and tri-partitions encode the same object. The former adopts a "vertical" point of view, whereas the latter adopts a "horizontal" one. This correspondence immediately implies several properties on both sides. For instance, consider two tri-partitions in dimensions $q+1$ and q, respectively. Then the boundary of the $(q+1)$-tree restrained to the q-cotree is invertible.

Another consequence is that perfect HDVFs can be mixed dimension-wise. More precisely, let X and $X' = (P', S', C')$ be two perfect HDVFs. Replacing the q-dimensional partition of X with (P'_q, S'_q, C'_q) still produces a perfect HDVF.

Last, let us point out that this equivalence prove that the definition of tri-partitions naturally extend to homology over a field, and not only for $\mathbb{Z}/2\mathbb{Z}$. Therefore, in what follows we consider homology computed over a field.

2.2.3. The canonical (co)cycle lemma

Another important consequence is that the canonical (co)cycle maps defined for tri-partitions (Proposition 2.2.2) coincide with those defined for HDVFs. Precisely, given a tri-partition (with maps $\tilde{z}_q$ and $\tilde{z}^q$) that is a layer of a given perfect HDVF (with maps z_q and z^q), we have $\tilde{z}_q = z_q$ and $\tilde{z}^q = z^q$. This follows directly from Propositions 2.2.4 and 2.2.5 together with Lemma 1.6.3.

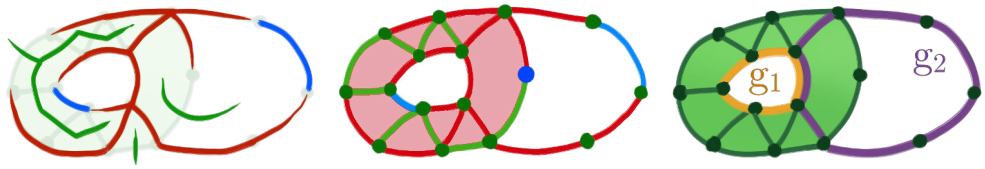


Figure 2.2 – A tri-partition (A^q, A_q, E_q) of dimension 1 (left), a perfect HDVF (P, S, C) (middle) and a 1-homology basis (right). The homology basis is the one induced by both the tri-partition and the HDVF, it is given by $\{\tilde{z}_q(\tau)\}_{\tau \in E_q}$ in terms of tri-partitions, and by $\{g_q(\tau)\}_{\tau \in C_q}$ in terms of HDVF. The canonical cycle lemma states that each generator is the unique cycle composed of a blue cell and red cells.

Historically, we realized the importance of the canonical (co)cycle maps after reading H. Edelsbrunner and Ölsböck 2020. Although this paper is more recent than the ones introducing HDVFs and their associated reductions, it makes clear that these maps generalize the reduction maps g and f^* in a meaningful way, for primary and secondary cells. This leads to the following central lemma, which has already been presented and proved in Section 1.6.1.

Lemma (canonical (co)cycle lemma, Lemma 1.6.3)**.** *In a perfect HDVF (P, S, C):*

$$\forall x \in K_q \qquad (x + \operatorname{Span} S) \cap \ker \partial = \{z_q(x)\} \qquad (x + \operatorname{Span} P) \cap \ker \partial^* = \{z^q(x)\}$$

In particular, if $c \in \operatorname{Span} C_q$ we have:

$$(c + \operatorname{Span} S) \cap \ker \partial = \{g_q(c)\} \qquad (c + \operatorname{Span} P) \cap \ker \partial^* = \{f_q^*(c)\}$$

This lemma is crucial because it provides a combinatorial characterization of (co)homology generators. More precisely, the homology generator associated to a critical cell γ is the unique cycle composed of γ and secondary cells (see Figure 2.1). Dually, the cohomology generator associated to γ is the unique cocycle composed of γ and primary cells. See Figure 1.19 for an illustration in Chapter 1. As a result, this lemma is used extensively in Section 2.3.

The behavior of the lemma on primary and secondary cells is also of interest. For example, in Section 3.2.3, we apply the lemma to primary cells to show that persistent homology generators are encoded in a single perfect HDVF.

All in all, the canonical (co)cycle lemma embodies a core aspect of our work: it highlights the interaction between the combinatorics of cells and the algebraic concept of homology.

We conclude with a related lemma providing a combinatorial characterization of the maps g and f^* for non-perfect HDVFs.

Lemma 2.2.6. *In a HDVF (P,S,C):*

$$\forall c \in \operatorname{Span} C_q \qquad (c + \operatorname{Span} S) \cap \ker(\mathtt{j}_P \partial) = \{g(c)\} \qquad (c + \operatorname{Span} P) \cap \ker(\mathtt{j}_S \partial^*) = \{f^*(c)\}$$

Proof. We only prove the first equality, as the other is obtained by duality. First, using Proposition 1.6.1, we have $g(c) \in c + \operatorname{Span} S$. Moreover, Definition 1.4.2 gives $\mathtt{j}_P \partial g = \mathtt{j}_P g d = 0$, because $\mathtt{j}_P g = 0$. Hence $g(c)$ also belongs to $(c + \operatorname{Span} S) \cap \ker(\mathtt{j}_P \partial)$.

It remains to prove the converse. Suppose that $x \in (c + \operatorname{Span} S) \cap \ker(\mathtt{j}_P \partial)$. We prove that $x - g(c) = 0$. As x and $g(c)$ are in $c + \operatorname{Span} S$, let us write $x = c + s$ and $g(c) = c + s'$. Then,

$$\begin{aligned} 0 &= \mathtt{j}_P \partial(x) - \mathtt{j}_P \partial(g(c)) = \mathtt{j}_P \partial(s - s') \\ &= \mathtt{j}_P \partial \mathtt{i}_S (s - s') && \text{because } s - s' \in \operatorname{Span} S \end{aligned}$$

Thus, as $\mathtt{j}_P \partial \mathtt{i}_S$ is invertible by definition of a HDVF, we have $s - s' = 0$, hence, $x = g(c)$. □

2.3. (Co)Homology Bases Computed by HDVF and Related Methods

A complex can admit a large number of possible homology and cohomology bases. It is known that computational homology approaches usually only compute a small subset of them. In particular, the standard approaches introduced earlier (see Section 1.4) can only produce elementary generators, i.e. cycles that are not the union of smaller cycles.

This is also the case for HDVFs and related methods. Some homology bases can be obtained using a perfect HDVF while others cannot, see Figure 2.4 for examples. Thus, this section provides a characterization of the homology and cohomology bases computed by HDVFs. This characterization highlights the importance of the role combinatorics plays in homology computations.

First, Section 2.3.1 introduces *explicit bases*, which can be understood as as a generalization of strictly fundamental bases for graphs to arbitrary combinatorial chain complexes. Then, Section 2.3.2 proves that a homology basis is explicit if and only if it

is induced by a HDVF. In particular, the proof provides an algorithm to compute a perfect HDVF given a specific explicit basis. Next, Section 2.3.3 establishes a similar result for cohomology bases, and Section 2.3.4 proves a necessary and sufficient condition under which a pair of explicit homology and cohomology bases can be computed by a single HDVF. Finally, we show in Section 2.3.5 how this result extends to other standard approaches, including for instance discrete Morse theory and standard persistent homology.

2.3.1. Explicit homology bases

In this subsection, we introduce *explicit homology bases* and their properties. The following results tightly intricate combinatorics (cells) and algebra (chains), which is consistent with the nature of HDVFs. Proposition 2.3.1 gives various equivalent definitions of explicit bases and Proposition 2.3.2 shows that the generators of an explicit basis are elementary.

Let us introduce the notion of subcomplex induced by a set of cells. First recall that given a chain x, $\overline{x}$ denotes its associated set of cells:

$$\overline{x} := \{\tau \in \overline{K}_q \mid \langle x, \tau \rangle \neq 0\}$$

Definition 2.3.1 (down induced complex)**.** Given a set of q-cells $A \subseteq \overline{K}_q$ in a complex (K, ∂), we denote $(K, \partial)^{\downarrow A}$ the subcomplex of (K, ∂) generated by the cells of A together with their $(q-1)$-faces. Formally, if we denote $\delta A := \cup_{\alpha \in A} \overline{\partial(\alpha)}$ the set of $(q-1)$-faces of cells of A, $(K, \partial)^{\downarrow A}$ is the following complex:

$$\{0\} \xrightarrow{0} \dots \xrightarrow{0} \operatorname{Span} A \xrightarrow{\partial_A^{\delta A}} \operatorname{Span} \delta A \xrightarrow{0} \{0\} \xrightarrow{0} \dots$$

Precisely, if we denote $(K', \partial') := (K, \partial)^{\downarrow A}$, we have $\overline{K'}_q = A$, $\overline{K'}_{q-1} = \delta A$ and $\overline{K'}_p = \emptyset$ for every other dimension p. In addition, as (K', ∂') is a subcomplex of (K, ∂), ∂' coincides with ∂, which gives $\partial'_q = \mathtt{j}_{K'_{q-1}} \circ \partial_q \circ \mathtt{i}_{K'_q} = \partial_A^{\delta A}$ and $\partial'_p = 0$ if $p \neq q$. See Figure 2.3 for an illustration.

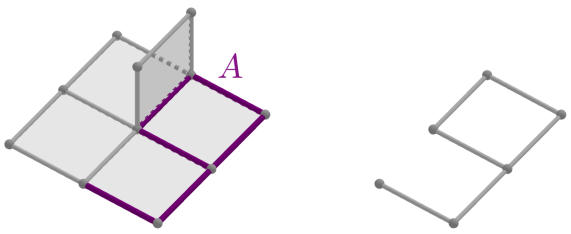


Figure 2.3 – A representation of the down induced complex of A. The left image represents a complex (K, ∂) and a set A of 1-cells (in purple). The right image represents the subcomplex $(K, \partial)^{\downarrow A}$ induced by the cells of A together with the cells in their boundary.

Remark. In the first version of this work (see GAZULL, GONZALEZ-LORENZO, et al. 2025a), we considered a slightly different concept. Precisely, instead of only considering the q and $q-1$ dimensions, we considered the complex closed downward for all lower dimensions (by taking the faces, the faces of the faces, etc.). Although both these choices yield coinciding definitions of *explicit* basis, we prefer to use Definition 2.3.1 as it is lighter, and eases some of the proofs (especially Proposition 2.3.7).

Let us fix (K,∂) a complex and simply denote $\beta_q := \beta_q(K)$ its q-th Betti number. For clarity, a collection of β_q objects is denoted by $\{a_i\}_{i\leq\beta_q}$ instead of $\{a_i\}_{i=1}^{\beta_q}$.

Given a q-homology basis $\{\mathrm{g}_i\}_{i\leq\beta_q}$ for a (K,∂) and $J\subseteq\{1,\dots,\beta_q\}$ a set of indices, we denote $(K^J,\partial) := (K,\partial)^{\downarrow\cup_{i\in J}\overline{\mathrm{g}_i}}$ the complex generated by the cells in the generators indexed by J[1]. We denote ι_q^J the homology map induced by the inclusion $(K^J,\partial)\subseteq(K,\partial)$. With this notation, we also denote $(K^{\downarrow\mathrm{g}},\partial) := (K^{\{1,\dots,\beta_q\}},\partial)$ the subcomplex induced by the whole basis (see Figure 2.4) and $\iota_q^{\downarrow\mathrm{g}}$ its associated homology map.

Definition 2.3.2 (Explicit homology basis)**.** A q-homology basis $\{\mathrm{g}_i\}_{i\leq\beta_q}$ for a complex (K,∂) is said to be *explicit* if it satisfies the following property:

$$\forall k,\ \overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i}\neq\emptyset \qquad \text{and} \qquad \beta_q\left(K^{\downarrow\mathrm{g}}\right)=\beta_q$$

In other words, a homology basis is explicit if and only if no generator is included in the others and the subcomplex induced by the generators has exactly the same number of q-holes as in the initial complex (see Figure 2.4 for examples of non explicit homology bases).

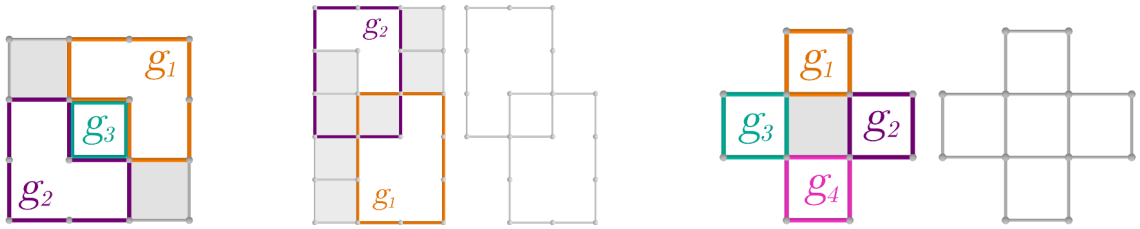


Figure 2.4 – Examples of 1-homology bases which are not explicit (right images: subcomplexes down induced by the basis). By Theorem 2.3.4 they cannot be obtained from a HDVF.
In (K_1), $\overline{\mathrm{g}_3}\setminus(\overline{\mathrm{g}_1}\cup\overline{\mathrm{g}_2})=\emptyset$ so the basis is not explicit. In (K_2), the complex $(K^{\downarrow\mathrm{g}},\partial)$ induced by the basis has three 1-holes instead of two: $3=\dim\left(H_1\left(K^{\downarrow\mathrm{g}}\right)\right)\neq\beta_1=2$. Similarly, in (K_3), the complex induced by the basis has five 1-holes instead of four.

The following proposition introduces equivalent definitions, which are useful in proofs of next subsections and next chapters.

[1] Note that K_q^J is the span of a combinatorial object $(\cup_{i\in J}\overline{\mathrm{g}_i})$; it is not necessarily equal to $\mathrm{Span}\left(\{\mathrm{g}_i\}_{i\in J}\right)$ (the "natural" algebraic object defined from generators).

Proposition 2.3.1. *Given a q-homology basis $\{\mathrm{g}_i\}_{i\leq\beta_q}$ for a complex (K,∂), the three following properties are equivalent:*

1. $\forall k,\ \overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i}\neq\emptyset$ *and* $\forall J,\ \iota_q^J : H_q\left(K^J\right)\to H_q(K)$ *is injective of rank* $|J|$
2. $\forall J\subseteq\{1,\dots,\beta_q\},\quad \ker\partial\cap K_q^J\subseteq\mathrm{Span}\left(\{\mathrm{g}_i\}_{i\in J}\right)$
3. $\forall k,\ \overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i}\neq\emptyset$ *and* $\beta_q\left(K^{\downarrow\mathrm{g}}\right)=\beta_q$

Proof. For simplicity, we write $[\cdot]$ instead of $[\cdot]_q^K$, $[\cdot]^J$ instead of $[\cdot]_q^{K^J}$ and $[\cdot]^{\downarrow\mathrm{g}}$ instead of $[\cdot]^{K^{\downarrow\mathrm{g}}}$. Firstly, we state some results about the q-homology basis $\{\mathrm{g}_i\}_{i\leq\beta_q}$:

- $\{[\mathrm{g}_i]\}_{i\leq\beta_q}$ is a basis of $H_q(K)$ (by Definition 1.3.11).
- $\iota_q^J([x]^J)=[x]$ for every x in K^J (by functoriality of the homology functor).
- For every J, $\{[\mathrm{g}_i]^J\}_{i\in J}$ is linearly independent (because $\{[\mathrm{g}_i]\}_{i\leq\beta_q}$ is linearly independent).

1. $\Longrightarrow$ 2. Let $x\in\ker\partial\cap K_q^J$, let us show that it is in $\mathrm{Span}\,\{\mathrm{g}_i\}_{i\in J}$

ι_q^J has rank $|J|$ and $\{[\mathrm{g}_i]\}_{i\in J}=\{\iota_q^J([\mathrm{g}_i]^J)\}_{i\in J}$ is a linearly independent set of cardinal $|J|$ in $\mathrm{im}\,\iota_q^J$, therefore $\{[\mathrm{g}_i]\}_{i\in J}$ is a basis of $\mathrm{im}\,\iota_q^J$. Hence, there exist $(\varepsilon_i)_{i\in J}$ coefficients such that:

$$\iota_q^J([x]^J)=\sum_{i\in J}\varepsilon_i[\mathrm{g}_i]=\iota_q^J\left(\left[\sum_{i\in J}\varepsilon_i\,\mathrm{g}_i\right]^J\right)$$

By injectivity of ι_q^J, we have $[x]^J=\left[\sum_{i\in J}\varepsilon_i\,\mathrm{g}_i\right]^J$, therefore there exists a $(q+1)$-chain $y\in K^J$ such that $x=\sum_{i\in J}\varepsilon_i\,\mathrm{g}_i+\partial y$. However, the maximal dimension of cells in K^J is q, so $y=0$. As a result, $x=\sum_{i\in J}\varepsilon_i\,\mathrm{g}_i\in\mathrm{Span}\,\{\mathrm{g}_i\}_{i\in J}$.

2. $\Longrightarrow$ 3. Firstly, by contradiction suppose that $\overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i}=\emptyset$.

Let $J:=\{1,\dots,\beta_q\}\setminus\{k\}$. We then have $\overline{\mathrm{g}_k}\subseteq\bigcup_{i\in J}\overline{\mathrm{g}_i}=\overline{K^J}$, therefore $\mathrm{g}_k\in K^J$. In addition, g_k is a q-cycle (because it is a q-generator). By assumption, we obtain that $\mathrm{g}_k\in\mathrm{Span}\,\{\mathrm{g}_i\}_{i\in J}$ which contradicts the fact that $\left\{[\mathrm{g}_i]^{\downarrow\mathrm{g}}\right\}_{i\leq\beta_q}$ is linearly independent. As a consequence $\overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i}\neq\emptyset$.

Then, $\left\{[\mathrm{g}_i]^{\downarrow\mathrm{g}}\right\}_{i\leq\beta_q}$ is linearly independent, let us show it is a basis for $H_q(K^{\downarrow\mathrm{g}})$. Let $[x]^{\downarrow\mathrm{g}}\in H_q(K^{\downarrow\mathrm{g}})$. We have that x is a q-cycle in $K^{\downarrow\mathrm{g}}$. By assumption we get that $x\in\mathrm{Span}\,\{\mathrm{g}_i\}_{i\leq\beta_q}$, therefore $[x]^{\downarrow\mathrm{g}}\in\mathrm{Span}\left\{[\mathrm{g}_i]^{\downarrow\mathrm{g}}\right\}_{i\leq\beta_q}$. Hence, $\left\{[\mathrm{g}_i]^{\downarrow\mathrm{g}}\right\}_{i\leq\beta_q}$ generates $H_q(K^{\downarrow\mathrm{g}})$ and is linearly independent so it is a basis for $H_q(K^{\downarrow\mathrm{g}})$. As a result $\dim(H_q(K^{\downarrow\mathrm{g}}))=\beta_q$.

3. $\Longrightarrow$ 1. Given J, $\left\{[\mathrm{g}_i]^J\right\}_{i\in J}$ is linearly independent, we first show that it is a basis for $H_q(K^J)$.

Consider a q-cycle x in K^J and its associated homology class $[x]^J\in H_q(K^J)$. Let $\iota_q^{J\to\downarrow\mathrm{g}}:H_q(K^J)\to H_q(K^{\downarrow\mathrm{g}})$ be the homology morphism induced by the inclusion $(K^J,\partial)\subseteq(K^{\downarrow\mathrm{g}},\partial)$. We have $\iota_q^{J\to\downarrow\mathrm{g}}([x]^J)=[x]^{\downarrow\mathrm{g}}$.

$H_q(K^{\downarrow \mathrm{g}})$ has dimension β_q by assumption. $\{[\mathrm{g}_i]^{\downarrow \mathrm{g}}\}_{i\leq\beta_q}$ is a linearly independent set of $H_q(K^{\downarrow \mathrm{g}})$ with cardinal β_q so it is a basis of it. Hence, there exist $(\varepsilon_i)_{i\in J}$ coefficients such that $[x]^{\downarrow \mathrm{g}} = \sum_{i\leq\beta_q} \varepsilon_i [\mathrm{g}_i]^{\downarrow \mathrm{g}}$. Thus, there exists a $(q+1)$-chain $y \in K^{\downarrow \mathrm{g}}_{q+1}$ such that $x = \sum_{i\leq\beta_q} \varepsilon_i \mathrm{g}_i + \partial y$. As $K^{\downarrow \mathrm{g}}_{q+1} = \{0\}$ we obtain $y = 0$ and $x = \sum_{i\leq\beta_q} \varepsilon_i \mathrm{g}_i$.

By contradiction: suppose that there is $j \notin J$ such that $\varepsilon_j \neq 0$. Then, as $\overline{\mathrm{g}_j} \setminus \bigcup_{i\neq j} \overline{\mathrm{g}_i} \neq \emptyset$, let γ be a cell in $\overline{\mathrm{g}_j}$ that is not in the other $\overline{\mathrm{g}_i}$. We have:

$$\langle x, \gamma\rangle = \sum_{i\leq\beta_q} \varepsilon_i \langle \mathrm{g}_i, \gamma\rangle = \varepsilon_j \langle \mathrm{g}_j, \gamma\rangle \neq 0 \qquad \text{because } \gamma \in \overline{\mathrm{g}_j} \text{ and } \varepsilon_j \neq 0$$

Hence, $\gamma \in \overline{x}$, which contradicts the fact that $x \in K^J$, because γ is not in any $\overline{\mathrm{g}_i}$ with $i \in J$. As a consequence, $\forall\, j \notin J, \varepsilon_j = 0$.

This implies that $x = \sum_{i\in J} \varepsilon_i \mathrm{g}_i$ and $[x]^J \in \mathrm{Span}\{[\mathrm{g}_i]^J\}_{i\in J}$.
As a result $\{[\mathrm{g}_i]^J\}_{i\in J}$ is a basis for $H_q(K^J)$ with cardinal $|J|$. Therefore, as $\{[\mathrm{g}_i]\}_{i\in J} = \{\iota^J_q([\mathrm{g}_i]^J)\}_{i\in J}$ is a linearly independent family of $\operatorname{im}\iota^J_q$ we obtain that ι^J_q is an injective map with rank $|J|$. □

Interestingly, all these definitions combine combinatorics (cells) and algebra (chains), but cannot be fully captured by either. Even 2. of Proposition 2.3.1 entails an intersection with K^J_q, whose definition relies on cells. This dual nature of bases computed by standard approaches is an important result of the present work. Moreover, we have the following combinatorial property:

Proposition 2.3.2. *The generators of an explicit homology basis are* elementary. *A cycle is said to be* elementary *if it is not composed of smaller cycles. Precisely, x is an* elementary *cycle if every cycle composed of cells of x is a multiple of x.*

Proof. Let $\{\mathrm{g}_i\}_{i\leq\beta_q}$ be an explicit q-homology basis of (K,∂) and a be a cycle such that $\overline{a} \subseteq \overline{\mathrm{g}_k}$. If we denote $J := \{k\}$, we have that a is in $\ker\partial \cap K^J_q$. By Proposition 2.3.1(2.), a is in $\mathrm{Span}(\mathrm{g}_k)$, so it is a multiple of g_k. □

This directly implies this additional property, which highlights the rigidity of explicit bases.

Proposition 2.3.3. *Let $\{\mathrm{g}_i\}_{i\leq\beta_q}$ and $\{\mathrm{g}'_i\}_{i\leq\beta_q}$ be explicit bases such that for all i, $\overline{\mathrm{g}_i} \subseteq \overline{\mathrm{g}'_i}$ or $\overline{\mathrm{g}'_i} \subseteq \overline{\mathrm{g}_i}$. Then, there exist coefficients $\lambda_i \in \mathscr{R}$ such that $\{\mathrm{g}_i\}_{i\leq\beta_q} = \{\lambda_i \mathrm{g}'_i\}_{i\leq\beta_q}$.*

Proof. By Proposition 2.3.2, generators are elementary, so $\overline{\mathrm{g}_i} \subseteq \overline{\mathrm{g}'_i}$ (as well as $\overline{\mathrm{g}'_i} \subseteq \overline{\mathrm{g}_i}$) implies that $\mathrm{g}_i = \lambda_i \mathrm{g}'_i$. □

2.3.2. Characterization of the homology bases computed by HDVFs

We now prove that the homology bases obtained from HDVFs are explicit.

Theorem 2.3.4. *Let $X = (P,S,C)$ be a perfect HDVF for (K,∂) with reduction (f,g,h). Then, for all q, the associated q-homology basis $\{g_q(\gamma)\}_{\gamma\in C_q}$ is explicit.*

Proof. Denote $C_q = \{\gamma_1,\dots,\gamma_{\beta_q}\}$. For this proof, we use the second characterization of an explicit homology basis in Proposition 2.3.1. Let $J \subseteq \{1,\dots,\beta_q\}$ and consider the down induced complex (K^J,∂). Let $x \in \ker\partial \cap K_q^J$, we will show that $x \in \operatorname{Span}\{g(\gamma_i)\}_{i\in J}$.

First, by Lemma 1.6.3, $g(\gamma_i)$ is a cycle composed of γ_i and secondary cells, thus, for all i, $\overline{g(\gamma_i)} \cap C = \{\gamma_i\}$. This implies:

$$\overline{K_q^J} \cap C = \bigcup_{i\in J} \overline{g(\gamma_i)} \cap C = \{\gamma_i,\ i \in J\}$$

Second, as $\operatorname{im} g \subseteq \operatorname{Span} S \oplus \operatorname{Span} C$ (see Proposition 1.6.1), we have that $\overline{x} \subseteq \overline{K_q^J} = \bigcup_{i\in J} \overline{g(\gamma_i)} \subseteq S \sqcup C$. Hence $x \in \operatorname{Span} S \oplus \operatorname{Span} C$ and there exists $s \in \operatorname{Span} S$ and $c \in \operatorname{Span} C$ such that $x = c + s$ and $\overline{x} = \overline{c} \sqcup \overline{s}$.

As $x \in K_q^J$, we obtain $\overline{x} \cap C = \overline{c} \subseteq \overline{K_q^J} \cap C = \{\gamma_i,\ i \in J\}$. This implies that there exist $(\varepsilon_i)_{i\in J}$ elements of the field $\mathscr{R}$ such that $c = \sum_{i\in J} \varepsilon_i \gamma_i$. Thus, with reduction properties (see Definition 1.4.2):

$$\begin{aligned}
x &= gf(x) + \partial h(x) + h\partial(x) \\
&= gf(c+s) + \partial h(x) && \text{as } x \in \ker\partial \\
&= gf(c+s) && \text{as } h \circ \mathtt{i}_{S\sqcup C} = 0 \text{ and } \overline{x} \in S \sqcup C \\
&= g(c) && \text{as } f \circ \mathtt{i}_C = id_C \text{ and } f \circ \mathtt{i}_S = 0 \\
&= \textstyle\sum_{i\in J} \varepsilon_i g(\gamma_i) && \text{As a result } x \in \operatorname{Span}\{g(\gamma_i)\}_{i\in J}.
\end{aligned}$$

□

As stated by the following theorem, the converse also holds, up to non-zero coefficients in front of generators:

Theorem 2.3.5. *Let $\{\mathrm{g}_i\}_{i\le\beta_q}$ be an explicit q-homology basis for (K,∂). There exist a perfect HDVF $X = (P,S,C)$ for (K,∂) with reduction (f,g,h), and non-zero coefficients $\lambda_i \in \mathscr{R}$, such that*

$$\{g_q(\gamma)\}_{\gamma\in C_q} = \{\lambda_i\, \mathrm{g}_i\}_{i\le\beta_q}$$

Remark. When $\mathscr{R} = \mathbb{Z}/2\mathbb{Z}$, every λ_is is equal to 1, and homology bases given by perfect HDVFs exactly correspond to explicit bases.

We prove Theorem 2.3.5 by building a perfect HDVF from an explicit basis. To do so, we need three lemmas related to the existence and completion of HDVFs.

Firstly, during a completion process, it is possible to preserve some critical cells. Precisely, it is possible to preserve every critical cell γ such that $g(\gamma)$ is a cycle that is not trivial in the homology group. This result is stated in the technical Lemma 1.6.13. In the case of a completion from a perfect HDVF for (K,∂) to a perfect HDVF for a

bigger complex $(K',\partial) \supseteq (K,\partial)$, Lemma 1.6.13 implies that it is possible to preserve the critical cells associated with holes of (K,∂) that were not filled in (K',∂).

Secondly, some critical cells preserved in a completion process keep their associated homology generators. This statement is a direct consequence of Lemma 1.6.15. Precisely, given $X \prec X'$ and γ a critical cell in both X and X', if the canonical cycle of γ in X is still a cycle in the larger complex, then it is equal to the canonical cycle of γ in X'.

Finally, given an explicit homology basis, it is possible to build a perfect HDVF X^{g} for the subcomplex $\left(K^{\downarrow \mathrm{g}},\partial\right)$ induced by the basis, such that X^{g} is still associated to the same basis:

Lemma 2.3.6 (perfect HDVF of the basis subcomplex)**.** *Let $\{\mathrm{g}_i\}_{i\leq\beta_q}$ be an explicit q-homology basis for (K,∂). Then, there exist $X^{\mathrm{g}} = (P^{\mathrm{g}}, S^{\mathrm{g}}, C^{\mathrm{g}})$ a perfect HDVF for $\left(K^{\downarrow \mathrm{g}},\partial\right)$ with reduction $(f^{\mathrm{g}}, g^{\mathrm{g}}, h^{\mathrm{g}})$, and non-zero coefficients λ_i, such that*

$$\left\{g^{\mathrm{g}}_q(\gamma)\right\}_{\gamma\in C^{\mathrm{g}}_q} = \{\lambda_i\, \mathrm{g}_i\}_{i\leq\beta_q}$$

The construction of this HDVF is technical and is done by induction on $k \in \{0,\dots\beta_q\}$.

Proof. For simplicity, we write K^k instead of $K^{\{1\dots k\}}$ and $[\cdot]^k$ instead of $[\cdot]^{K^k}_q$. In particular, $K^{\beta_q} = K^{\downarrow \mathrm{g}}$. We also write ι^k instead of $\iota^{K^k}_q$ and omit the q subscripts for reduction maps.

Let $\{\mathrm{g}_i\}_{i\leq\beta_q}$ be an explicit q-homology basis for (K,∂). Let $k \leq \beta_q$. Because $\overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i} \neq \emptyset$, let us fix an element γ_k in $\overline{\mathrm{g}_k}$ that does not belong to the other $\overline{\mathrm{g}_i}$s.

We build by induction for $k = 0,\dots,\beta_q$ a perfect HDVF $X^k = (P^k, S^k, C^k)$ for the complex (K^k,∂) (with reduction (f^k, g^k, h^k)) such that $C^k_q = \{\gamma_i,\ i \leq k\}$ and $\forall i \leq k,\ g^k(\gamma_i) = \lambda_i\, \mathrm{g}_i$ (with λ_i non zero coefficients). Let us recall the reduction diagram:

$$\begin{array}{ccccccccc}
\dots & \xrightarrow[h^k_q]{\partial^k_{q+1}} & K^k_q & \xrightarrow[h^k_{q-1}]{\partial^k_q} & K^k_{q-1} & \xrightarrow[h^k_{q-2}]{\partial^k_{q-1}} & \dots \\
 & & {\scriptstyle g^k_q}\uparrow\downarrow{\scriptstyle f^k_q} & & {\scriptstyle g^k_{q-1}}\uparrow\downarrow{\scriptstyle f^k_{q-1}} & & \\
\dots & \xrightarrow{0} & \operatorname{Span} C^k_q & \xrightarrow{0} & \operatorname{Span} C^k_{q-1} & \xrightarrow{0} & \dots
\end{array}$$

Base case: $k = 0$.
$K^0 = \{0\}$, $X^0 = (\emptyset,\emptyset,\emptyset)$ with reduction (f^0, g^0, h^0), such that f^0, g^0 and h^0 are the zero function $\{0\} \to \{0\}$.

Induction step: $k+1$.
Suppose that there exist $X^k = (P^k, S^k, C^k)$ a perfect HDVF for the complex (K^k,∂) (with reduction (f^k, g^k, h^k)), such that $C^k_q = \{\gamma_i\}_{i\leq k}$ and $\forall i \leq k,\ g^k(\gamma_i) = \lambda_i\, \mathrm{g}_i$. The proof follows the following steps:

1. We build a perfect HDVF X for K^{k+1} such that $X^k \prec X$.
2. We modify X with a `W` to get a perfect HDVF X^{k+1} for K^{k+1} such that $X^k \prec X^{k+1}$.

3. We show that X^{k+1} satisfies the required properties.

1. We have $\overline{K^{k+1}} = \overline{K^k} \cup \overline{\mathrm{g}_{k+1}}$. Denote ι^k, ι^{k+1} and $\iota^{k\to k+1}$ the following q-homology maps induced by the complex inclusions $(K^k,\partial^k) \subseteq (K,\partial)$, $(K^{k+1},\partial^{k+1}) \subseteq (K,\partial^k)$ and $(K^k,\partial^k) \subseteq (K^{k+1},\partial^{k+1})$:

$$\iota^k : H_q(K^k) \to H_q(K) \quad \iota^{k+1} : H_q(K^{k+1}) \to H_q(K) \quad \iota^{k\to k+1} : H_q(K^k) \to H_q(K^{k+1})$$

By functoriality of the homology functor we have $\iota^k = \iota^{k+1} \circ \iota^{k\to k+1}$.

$\{\mathrm{g}_i\}_{i\le\beta_q}$ is explicit so ι^k is injective by Proposition 2.3.1(1.), therefore $\iota^{k\to k+1}$ is also injective. In addition, $\left\{[g^k(\gamma)]^k\right\}_{\gamma\in C_q^k}$ is linearly independent (because X^k is perfect). Hence, $\left\{[g^k(\gamma)]^{k+1}\right\}_{\gamma\in C_q^k} = \left\{\iota^{k\to k+1}[g^k(\gamma)]^k\right\}_{\gamma\in C_q^k}$ is linearly independent in $H_q(K^{k+1})$.

Using Lemma 1.6.13 with $C^* := C_q^k$, it is then possible to complete X^k into a perfect HDVF $X = (P,S,C)$ for (K^{k+1},∂^{k+1}), such that $P^k \subseteq P$ and $S^k \subseteq S$ and $C_q^k \subseteq C_q$.

2. We now slightly modify X to have the desired properties.
$(\mathrm{g}_i)_{i\le\beta_q}$ is explicit so ι^{k+1} is injective with rank $k+1$ by Proposition 2.3.1, therefore $|C_q| = k+1$. As $|C_q^k| = k$ there exists a cell $\tau \in C_q$ such that $C_q = C_q^k \sqcup \{\tau\}$. As τ is not in $\overline{K^k}$, we have that $\tau \in \overline{\mathrm{g}_{k+1}}$.

The maximal dimension of cells in the complex (K^{k+1},∂) is q by construction, therefore every q-cell of it is either a secondary or a critical cell (otherwise ∂_S^P would not be invertible). Hence, we have that $\overline{\mathrm{g}_{k+1}}\backslash\{\tau\} \subseteq S$ because $\overline{\mathrm{g}_{k+1}}$ cannot contain critical cells from $C_q^k = \{\gamma_i,\ i\le k\}$. This implies that $\mathrm{g}_{k+1} \in \langle \mathrm{g}_{k+1},\tau\rangle \cdot \tau + \operatorname{Span} S$, with $\langle \mathrm{g}_{k+1},\tau\rangle \neq 0$. As a result, by Lemma 1.6.3 we have $g(\tau) = \mu_{k+1}\,\mathrm{g}_{k+1}$, with $\mu_{k+1} := \langle \mathrm{g}_{k+1},\tau\rangle^{-1}$ a non-zero coefficient.

If $\tau = \gamma_{k+1}$, we can just define $X^{k+1} := X$. If that is not the case we use a $\mathtt{W}$ operation: as $\gamma_{k+1} \in \overline{\mathrm{g}_{k+1}} = \overline{g(\tau)}$, $\mathtt{W}^{\{\tau\}}_{\{\gamma_{k+1}\}}$ is a valid operation for X (see Definition 1.6.3). As a result we define $X^{k+1} := \mathtt{W}^{\{\tau\}}_{\{\gamma_{k+1}\}} X = (P^{k+1},S^{k+1},C^{k+1})$, which is a perfect HDVF for K^{k+1} because a valid $\mathtt{W}$ operation transforms a perfect HDVF into another perfect HDVF(see Definition 1.6.3). We denote $(f^{k+1},g^{k+1},h^{k+1})$ its reduction.

3. We now verify that X^{k+1} satisfies the desired properties.
Firstly, $C_q^{k+1} = \{\gamma_i,\ i\le k+1\}$. Secondly, as $P^k \subseteq P^{k+1}$, $S^k \subseteq S^{k+1}$ and $C_q^k \subseteq C \cap C^{k+1}$, Lemma 1.6.15 implies that $\forall i \le k$, $g^{k+1}(\gamma_i) = g^k(\gamma_i) = \lambda_i\,\mathrm{g}_i$.
In addition we prove that we also have $g^{k+1}(\gamma_{k+1}) = \lambda_{k+1}\,\mathrm{g}_{k+1}$, with $\lambda_{k+1} := \langle \mathrm{g}_{k+1},\gamma_{k+1}\rangle$ a non-zero coefficient. τ and γ_{k+1} are both cells of $\mu_{k+1}\,\mathrm{g}_{k+1} = g(\tau)$, which is a chain in $\tau + \operatorname{Span} S$. Hence, there exists $s \in \operatorname{Span} S$ such that $\mathrm{g}_{k+1} = \lambda_{k+1}\cdot\gamma_{k+1} + \mu_{k+1}^{-1}\cdot\tau + s$. We have $S^{k+1} = S \sqcup \{\tau\}\backslash\{\gamma_{k+1}\}$, so $\mathrm{g}_{k+1} \in \lambda_{k+1}\gamma_{k+1} + \operatorname{Span} S^{k+1}$. By Lemma 1.6.3 we get $g^{k+1}(\gamma_{k+1}) = \lambda_{k+1}\,\mathrm{g}_{k+1}$. □

Once previous lemmas are known, the proof of Theorem 2.3.5 becomes intuitive and Figure 2.5 illustrates it.

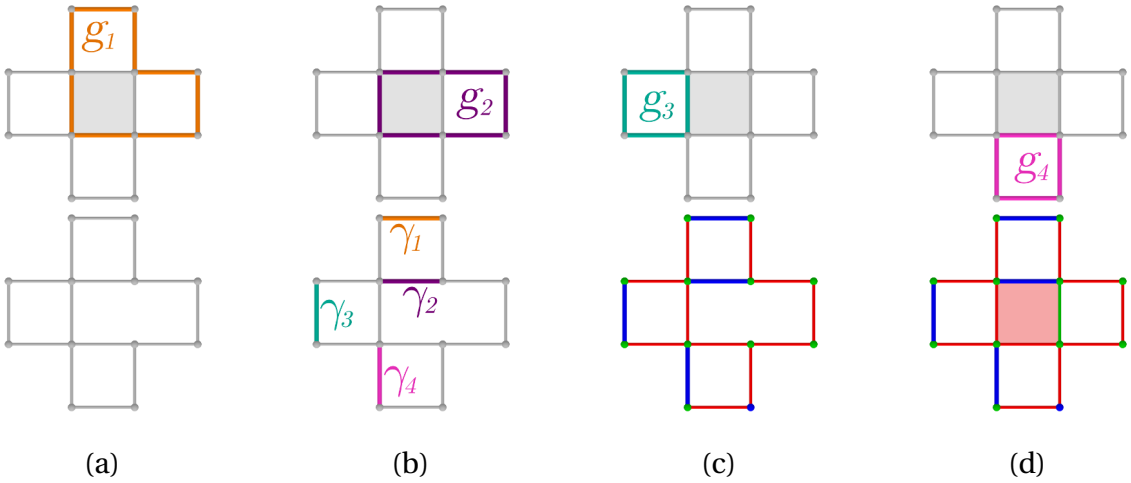


Figure 2.5 – Illustration of the constructions involved in the proof of Theorem 2.3.5. The first row represents four generators forming an explicit 1-homology basis of a complex (K,∂). The second row shows steps of the proof. It starts (a) with $(K^{\downarrow \mathrm{g}},\partial)$ the complex induced by this basis (which has four 1-holes as the basis is explicit). (b) represents a choice of cells γ_k that are respectively in $\overline{\mathrm{g}_k}\setminus\bigcup_{i\neq k}\overline{\mathrm{g}_i}$ (which is non-empty as the basis is explicit). (c) is the perfect HDVF X^{g} obtained with Lemma 2.3.6. By construction, we get $g^{\mathrm{g}}(\gamma_i)=\mathrm{g}_i$. Finally, (d) is X the perfect HDVF for (K,∂) obtained by completing X^{g} using Lemma 1.6.13.
The 1-homology basis induced by X is $\{\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3,\mathrm{g}_4\}$, by Lemma 1.6.15.

Proof. [*Proof of Theorem 2.3.5*] Using Lemma 2.3.6, there exists $X^{\mathrm{g}}=(P^{\mathrm{g}},S^{\mathrm{g}},C^{\mathrm{g}})$ a perfect HDVF for $(K^{\downarrow \mathrm{g}},\partial)$ (with reduction $(f^{\mathrm{g}},g^{\mathrm{g}},h^{\mathrm{g}})$) such that its associated q-homology basis is $\{\lambda_i\,\mathrm{g}_i\}_{i\leq\beta_q}$, with λ_i non-zero coefficients.

As $\{\mathrm{g}_i\}_{i\leq\beta_q}$ is explicit, we have that $\iota_q^{\downarrow \mathrm{g}}: H_q(K^{\downarrow \mathrm{g}})\to H_q(K)$ is injective, see Proposition 2.3.1(1). $\left\{[g^{\mathrm{g}}(\gamma)]^{K^{\downarrow \mathrm{g}}}\right\}_{\gamma\in C_q^{\mathrm{g}}}$ is linearly independent (because X^{g} is perfect), so we obtain that $\left\{\left[g_q^{\mathrm{g}}(\gamma)\right]^{K}\right\}_{\gamma\in C_q^{\mathrm{g}}}=\left\{\iota_q^{\downarrow \mathrm{g}}\left[g_q^{\mathrm{g}}(\gamma)\right]^{K^{\downarrow \mathrm{g}}}\right\}_{\gamma\in C_q^{\mathrm{g}}}$ is linearly independent in $H_q(K)$.

As a consequence, we use Lemma 1.6.13 with $C^*:=C_q^{\mathrm{g}}$ to complete X^{g} into a perfect HDVF $X=(P,S,C)$ for (K,∂), with reduction (f,g,h), such that $X^{\mathrm{g}}\prec X$ and $C_q^{\mathrm{g}}\subseteq C_q$. As $\beta_q=\beta_q(K)=|C_q|$ and $\beta_q=\beta_q(K^{\downarrow \mathrm{g}})=|C_q^{\mathrm{g}}|$ we get $C_q^{\mathrm{g}}=C_q$.

Every generator $\mathrm{g}_i=g_q^{\mathrm{g}}(\gamma_i)$ is a cycle in (K,∂), so we can use Lemma 1.6.15 with $X^{\mathrm{g}}\prec X$ to obtain $\{g_q(\gamma)\}_{\gamma\in C_q}=\left\{g_q^{\mathrm{g}}(\gamma)\right\}_{\gamma\in C_q^{\mathrm{g}}}=\{\lambda_i\,\mathrm{g}_i\}_{i\leq\beta_q}$. □

As a result, Theorems 2.3.4 and 2.3.5 imply that a homology basis is explicit if and only if it is induced by a perfect HDVF (up to non-zero coefficients). Note that the proof of Theorem 2.3.5 provides an algorithm to build a perfect HDVF given an explicit homology basis.

2.3.3. Explicit cohomology bases

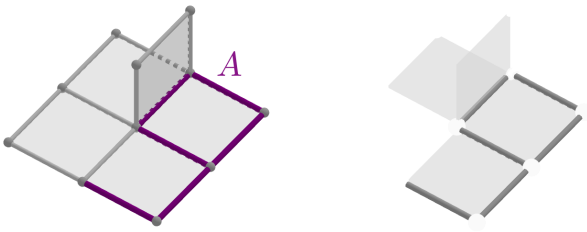
In this subsection, we use duality and results from Section 2.3 to characterize the cohomology bases computed by HDVFs. In the next subsection, we investigate whether a pair of homology and cohomology bases is compatible, i.e. if both bases can be induced simultaneously by a single HDVF.

Let us first define the notion of *explicit* cohomology basis using the notion of up induced complex, which is the dual of the down induced complex. We then prove that a cohomology basis is explicit if and only if it is an explicit homology basis of the dual complex, see Proposition 2.3.7.

Definition 2.3.3 (up induced complex)**.** Given a set of q-cells $A \subseteq \overline{K}_q$ in a complex (K,∂), we denote $(K,\partial)^{\uparrow A}$ the subcomplex of (K,∂) generated by the cells of A together with their $(q+1)$-cofaces. Formally, if we denote $\delta^* A := \cup_{\alpha\in A}\overline{\partial^*(\alpha)}$ the $(q+1)$-cofaces of A, $(K,\partial)^{\uparrow A}$ is the following complex:

$$\{0\} \xrightarrow{0} \dots \xrightarrow{0} \operatorname{Span}\delta^* A \xrightarrow{\partial^A_{\delta^* A}} \operatorname{Span} A \xrightarrow{0} \dots \xrightarrow{0} \{0\}$$

$(K,\partial)^{\downarrow A}$, $(K,\partial)^{\uparrow A}$ and $(K,\partial)^*$ are well-defined combinatorial chain complexes, but it can sometimes be hard to intuitively and visually understand them. This is because these complexes may not be derived from standard geometric complexes (e.g. simplicial or cubical complexes), even when it is the case for (K,∂). For instance, there can be edges in $(K,\partial)^{\uparrow A}$ that do not have any vertices in their boundary. See Figure 2.6 for an illustration.



(K,∂) $\qquad$ $(K,\partial)^{\uparrow A}$

Figure 2.6 – Representations of the up induced complex. The left image represents a complex (K,∂) and a set A of 1-cells (in purple). The right image represents the subcomplex $(K,\partial)^{\uparrow A}$ induced by the cells of A together with the cells in their coboundary.

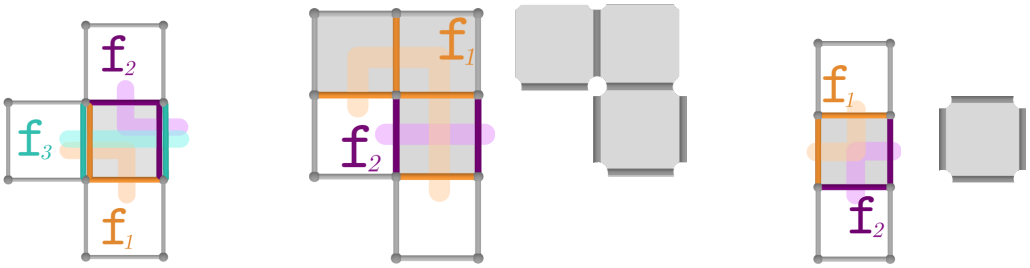
Given a q-cohomology basis $\{\mathtt{f}_i\}_{i\leq\beta_q}$ for a complex (K,∂), we denote $(K^{\uparrow\mathtt{f}},\partial) = (K,\partial)^{\uparrow\cup_{i\leq\beta_q}\overline{\mathtt{f}_i}}$ the subcomplex induced by the cells of all generators $\mathtt{f}_i$ and their $(q+1)$-cofaces.

Definition 2.3.4 (Explicit cohomology bases)**.** A q-cohomology basis $\{\mathtt{f}_i\}_{i\leq\beta_q}$ for a

complex (K,∂) is said to be *explicit* if it satisfies the following property:

$$\forall k,\ \overline{\mathtt{f}_k} \setminus \bigcup_{i\neq k} \overline{\mathtt{f}_i} \neq \emptyset \qquad \text{and} \qquad \beta^q\left(K^{\uparrow \mathtt{f}}\right) = \beta_q$$

See Figure 2.7 for an illustration of cohomology bases that are not explicit.



(K_1) (K_2) (K_3)

Figure 2.7 – Examples of 1-cohomology bases which are not explicit (right images: subcomplexes up induced by the basis). By Theorem 2.3.8 they cannot be obtained from a HDVF. In (K_1), $\overline{\mathtt{f}_3}$ is included in $\overline{\mathtt{f}_1} \cup \overline{\mathtt{f}_2}$, so the basis is not explicit. In (K_2), the complex $(K^{\uparrow \mathtt{f}},\partial)$ induced by the basis (on the right) has a 1-cohomology group of dimension 3, but K_2 has only two 1-holes. Thus, the basis is not explicit. Similarly, in (K_3), $H^1\left(K^{\uparrow \mathtt{g}}\right)$ has dimension 3 instead of $2 = \beta_1$.
Note that, when considering geometrical complexes, it is harder to intuitively count the holes for an up induced complex than for a down induced complex (see Figure 2.4).

Proposition 2.3.7. *A q-cohomology basis $\{\mathtt{f}_i\}_{i\leq\beta_q}$ for a complex (K,∂) is explicit if and only if it is an explicit $(n-q)$-homology basis of the dual complex $(K,\partial)^*$.*

Proof. Let A be a subset of the q-cells of (K,∂). We first prove the equality between the complexes $((K,\partial)^{\uparrow A})^*$ and $((K,\partial)^*)^{\downarrow A}$. Using Definitions 1.3.12 and 2.3.3 we have the following complexes:

$$(K,\partial)^{\uparrow A}: \quad \{0\} \xrightarrow{0} \dots \xrightarrow{0} \operatorname{Span}\delta^* A \xrightarrow{\mathtt{j}_A \circ \partial \circ \mathtt{i}_{\delta^* A}} \operatorname{Span} A \xrightarrow{0} \dots \xrightarrow{0} \{0\}$$
$$((K,\partial)^{\uparrow A})^*: \quad \{0\} \xleftarrow{0} \dots \xleftarrow{0} \operatorname{Span}\delta^* A \xleftarrow{(\mathtt{j}_A \circ \partial \circ \mathtt{i}_{\delta^* A})^*} \operatorname{Span} A \xleftarrow{0} \dots \xleftarrow{0} \{0\}$$

Moreover, using Definitions 1.3.12 and 2.3.1 we obtain the following:

$$(K,\partial)^*: \quad \{0\} \xleftarrow{\partial^*_{n+1}} K_n \xleftarrow{\partial^*_n} \dots \xleftarrow{\partial^*_{q+2}} K_{q+1} \xleftarrow{\partial^*_{q+1}} K_q \xleftarrow{\partial^*_q} \dots \xleftarrow{\partial^*_1} K_0$$
$$((K,\partial)^*)^{\downarrow A}: \quad \{0\} \xleftarrow{0} \dots \xleftarrow{0} \operatorname{Span}\delta^* A \xleftarrow{\mathtt{j}_{\delta^* A} \circ \partial^* \circ \mathtt{i}_A} \operatorname{Span} A \xleftarrow{0} \dots \xleftarrow{0} \{0\}$$

As $(\mathtt{j}_A \circ \partial \circ \mathtt{i}_{\delta^* A})^* = \mathtt{j}_{\delta^* A} \circ \partial^* \circ \mathtt{i}_A$, we obtain $((K,\partial)^{\uparrow A})^* = ((K,\partial)^*)^{\downarrow A}$.

Thus, as the q-cohomology of $(K,\partial)^{\uparrow A}$ is the $(n-q)$-homology of $((K,\partial)^{\uparrow A})^*$ we obtain the following:

$$H^q\big((K,\partial)^{\uparrow A}\big) = H_{n-q}\big(((K,\partial)^*)^{\downarrow A}\big)$$

Now, using $A = \bigcup_{i\leq\beta_q} \overline{\mathtt{f}_i}$, we get that $H^q\left((K,\partial)^{\uparrow \mathtt{f}}\right)$ and $H_{n-q}\left(((K,\partial)^*)^{\downarrow \mathtt{f}}\right)$ are equal. As a result, $\{\mathtt{f}_i\}_{i\leq\beta_q}$ is a cohomology explicit basis for (K,∂) if and only if it is a $(n-q)$-homology explicit basis of $(K,\partial)^*$. □

Propositions 1.6.5 and 2.3.7 and Theorems 2.3.4 and 2.3.5 imply the following characterization of cohomology bases:

Theorem 2.3.8. *If (P,S,C) is a perfect HDVFs for (K,∂) with reduction (f,g,h), then $\{f_q^*(\gamma)\}_{\gamma\in C_q}$ is an explicit cohomology basis. Conversely, given $\{\mathtt{f}_i\}_{i\leq\beta_q}$ an explicit q-cohomology basis for a complex (K,∂), there exist a perfect HDVF (P,S,C) for (K,∂) with reduction (f,g,h), and non-zero coefficients λ_i, such that*

$$\{f_q^*(\gamma)\}_{\gamma\in C_q} = \{\lambda_i\,\mathtt{f}_i\}_{i\leq\beta_q}$$

2.3.4. Combining explicit homology and cohomology bases

As previous subsections characterized homology and cohomology bases computed by HDVF, a natural question arises: given an explicit q-homology basis $\{\mathtt{g}_i\}_{i\leq\beta_q}$ together with an explicit q-cohomology basis $\{\mathtt{f}_i\}_{i\leq\beta_q}$, is it possible to build a perfect HDVF computing these bases simultaneously?

It turns out it is not always possible, as shown in Figure 2.8. Propositions 2.3.9 and 2.3.10 state a necessary and sufficient condition for the pair $(\{\mathtt{g}_i\}_{i\leq\beta_q},\{\mathtt{f}_i\}_{i\leq\beta_q})$ to be computable by a single HDVF.



Figure 2.8 – Both figures display a pair $(\{\mathtt{g}_i\}_{i\leq\beta_q},\{\mathtt{f}_i\}_{i\leq\beta_q})$ of homology and cohomology explicit bases that cannot be induced by a single HDVF. Indeed, in both cases, $\overline{\mathtt{g}_1}$ is intersecting more than once the cells of the cohomology generators.

Proposition 2.3.9. *Let $X = (P,S,C)$ be a perfect HDVF with reduction (f,g,h). Then, for all $\gamma\in C_q$, $\overline{g(\gamma)}\cap\overline{f^*(\gamma)} = \{\gamma\}$. Moreover, $\overline{g(\gamma)}\cap\overline{f^*(\gamma')} = \emptyset$ whenever $\gamma'\in C_q$ is different from γ.*

Proof. By using the fact that $g(\gamma) \in \gamma + \operatorname{Span} S$ and $f^*(\gamma) \in \gamma + \operatorname{Span} P$ (see Lemma 1.6.3), we obtain the desired properties. □

Proposition 2.3.10. *Let $\{\mathtt{g}_i\}_{i\leq\beta_q}$ and $\{\mathtt{f}_i\}_{i\leq\beta_q}$ be explicit q-homology and q-cohomology bases, respectively. Suppose that, up to a permutation of the indices, for all i, there exists a cell γ_i such that $\overline{\mathtt{g}_i} \cap \overline{\mathtt{f}_i} = \{\gamma_i\}$. Moreover, suppose that for all $j \neq i$, $\overline{\mathtt{g}_i} \cap \overline{\mathtt{f}_j} = \emptyset$. Then, there exists a perfect HDVF X inducing both $\{\mathtt{g}_i\}_{i\leq\beta_q}$ and $\{\mathtt{f}_i\}_{i\leq\beta_q}$ (up to non-zero coefficients).*

The proof of this proposition is constructive: we build a suited HDVF using the constructions involved in the proof of Theorem 2.3.5. This construction process is illustrated in Figure 2.9.

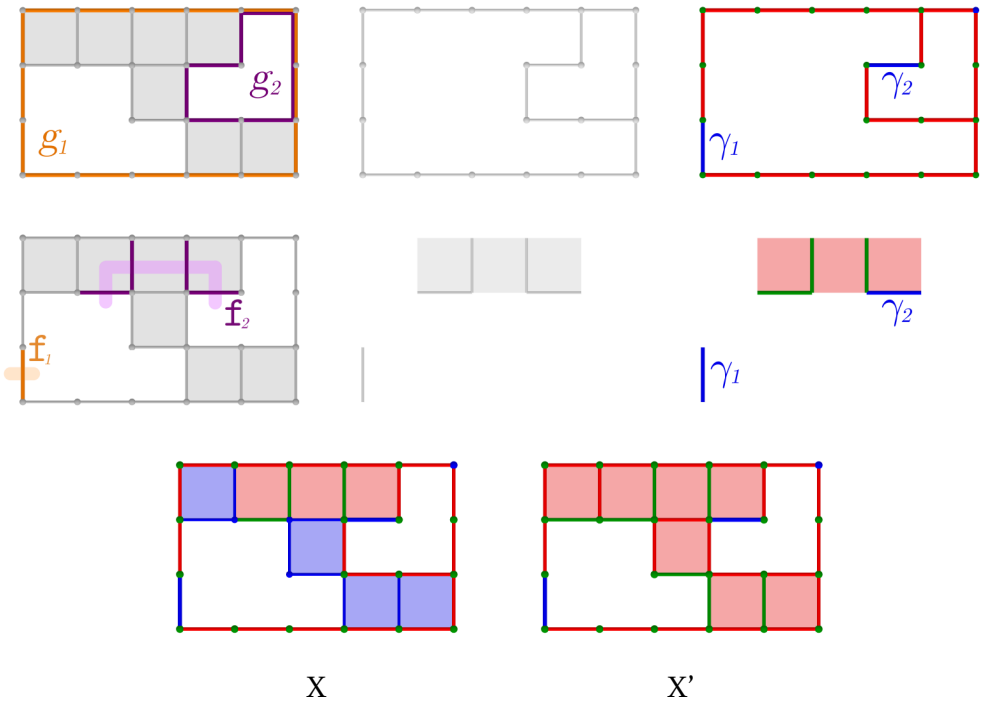


Figure 2.9 – Illustration of the construction in the proof of Proposition 2.3.10 to build a perfect HDVF that induces a given pair of compatible explicit homology and cohomology basis. First row: an explicit 1-homology basis $\{\mathtt{g}_1, \mathtt{g}_2\}$, its down induced complex $(K^{\downarrow \mathtt{g}}, \partial)$, and its associated HDVF $X^{\mathtt{g}}$. Second row: an explicit 1-cohomology basis $\{\mathtt{f}_1, \mathtt{f}_2\}$ (compatible with $\{\mathtt{g}_1, \mathtt{g}_2\}$), its up induced complex $(K^{\uparrow \mathtt{f}}, \partial)$, and its associated HDVF $X^{\mathtt{f}}$. Third row: the HDVF X, obtained by combining $X^{\mathtt{g}}$ and $X^{\mathtt{f}}$, and its perfect completion X', obtained by preserving the critical cells γ_1 and γ_2 using Lemma 1.6.13.

Proof. Again, for simplicity, we do the proof with the field $\mathscr{R} = \mathbb{Z}/2\mathbb{Z}$ to avoid writing the non-zero coefficients.

First, every γ_i is in $\overline{\mathtt{g}_i}$ but not in the other $\overline{\mathtt{g}_j}$, otherwise we would have $\gamma_i \in \overline{\mathtt{f}_i} \cap \overline{\mathtt{g}_j} = \emptyset$ for $i \neq j$. Similarly, $\gamma_i \in \overline{\mathtt{f}_i} \setminus \bigcup_{j \neq i} \overline{\mathtt{f}_j}$.

Using Lemma 2.3.6, we can build a perfect HDVF $X^{\mathtt{g}} = (P^{\mathtt{g}}, S^{\mathtt{g}}, C^{\mathtt{g}})$ for $K^{\downarrow \mathtt{g}}$, with reduction $(f^{\mathtt{g}}, g^{\mathtt{g}}, h^{\mathtt{g}})$, such that $\{\gamma_i\}_{i \leq \beta_q} = C^{\mathtt{g}}_q$ and $\forall i \leq \beta_q,\ g^{\mathtt{g}}(\gamma_i) = \mathtt{g}_i$. Dually, it is possible to build a perfect HDVF $X^{\mathtt{f}} = (P^{\mathtt{f}}, S^{\mathtt{f}}, C^{\mathtt{f}})$ for $K^{\uparrow \mathtt{f}}$, with reduction $(f^{\mathtt{f}}, g^{\mathtt{f}}, h^{\mathtt{f}})$, such that $\{\gamma_i\}_{i \leq \beta_q} = C^{\mathtt{f}}_q$ and $\forall i \leq \beta_q,\ f^{\mathtt{f}^*}(\gamma_i) = \mathtt{f}_i$. To be precise, we can build $X^{\mathtt{f}}$ using Lemma 2.3.6 on $(K, \partial)^*$, and then using Propositions 1.6.5 and 2.3.7.

We can then define $X = (P, S, C)$ as the union of $X^{\mathtt{g}}$ and $X^{\mathtt{f}}$, considered in the whole complex (K, ∂):

$$P := P^{\mathtt{g}} \cup P^{\mathtt{f}} \qquad S := S^{\mathtt{g}} \cup S^{\mathtt{f}} \qquad C := \overline{K} \setminus (P \cup S)$$

Let us prove that X is a HDVF:

- First, we show that X is a partition of $\overline{K}$.
  The complex $K^{\downarrow \mathtt{g}}$ has maximal dimension q, whereas the complex $K^{\uparrow \mathtt{f}}$ has minimal dimension q, hence:
  $$\text{if } p > q, \text{ then } P^{\mathtt{g}}_p = S^{\mathtt{g}}_p = \emptyset \qquad \text{if } p < q, \text{ then } P^{\mathtt{f}}_p = S^{\mathtt{f}}_p = \emptyset$$
  As a consequence, if $p \neq q$, (P_p, S_p) is equal to either $(P^{\mathtt{g}}_p, S^{\mathtt{g}}_p)$ or $(P^{\mathtt{f}}_p, S^{\mathtt{f}}_p)$. In both cases, (P_p, S_p, C_p) is a partition of the p-cells, as $P^{\mathtt{g}}_p \cap S^{\mathtt{g}}_p = \emptyset$ and $P^{\mathtt{f}}_p \cap S^{\mathtt{f}}_p = \emptyset$.
  Now consider the dimension q. We have $P^{\mathtt{g}}_q = \emptyset$ as it is in bijection with $S^{\mathtt{g}}_{q+1} = \emptyset$ (as $\mathtt{j}_{P_q} \circ \partial \circ \mathtt{i}_{S_{q+1}}$ is invertible). Similarly, $S^{\mathtt{f}}_q = \emptyset$. Thus, we obtain $(P_q, S_q) = (P^{\mathtt{f}}_q, S^{\mathtt{g}}_q)$.
  By assumptions, $\bigcup_{i \leq \beta_q} \overline{\mathtt{g}_i}$ and $\bigcup_{i \leq \beta_q} \overline{\mathtt{f}_i}$ only intersect on the cells $\{\gamma_i\}_{i \leq \beta_q}$, which are critical in both $X^{\mathtt{g}}$ and $X^{\mathtt{f}}$. Therefore, $P^{\mathtt{f}}_q \cap S^{\mathtt{g}}_q = \emptyset$, otherwise there would be a cell γ_i that is either both secondary and critical in $X^{\mathtt{g}}$ or both primary and critical in $X^{\mathtt{f}}$. Thus, (P_q, S_q, C_q) is a partition of the q-cells.
- Second, we show that X satisfies the HDVF invertibility condition.
  For $p < q$ we have $\mathtt{j}_{P_p} \circ \partial \circ \mathtt{i}_{S_{p+1}} = \mathtt{j}_{P^{\mathtt{g}}_p} \circ \partial \circ \mathtt{i}_{S^{\mathtt{g}}_{p+1}}$, which is invertible because $X^{\mathtt{g}}$ is a HDVF for a subcomplex of K. Similarly, for $p \geq q$ we have $\mathtt{j}_{P_p} \circ \partial \circ \mathtt{i}_{S_{p+1}} = \mathtt{j}_{P^{\mathtt{f}}_p} \circ \partial \circ \mathtt{i}_{S^{\mathtt{f}}_{p+1}}$, which is invertible because $X^{\mathtt{f}}$ is a HDVF for a subcomplex of K.

Finally, we complete X to obtain a perfect HDVF X' that induces both bases. We denote (f, g, h) the reduction associated with X. First, we need to show that for all i, $g(\gamma_i) = g^{\mathtt{g}}(\gamma_i)$ and $f^*(\gamma_i) = f^{\mathtt{f}^*}(\gamma_i)$. Using Proposition 1.6.1, we have:

$$g^{\mathtt{g}}(\gamma_i) = \gamma_i - H^{\mathtt{g}}_{q-1} \circ \mathtt{j}_{P^{\mathtt{g}}_{q-1}} \circ \partial(\gamma_i) \qquad g(\gamma_i) = \gamma_i - H_{q-1} \circ \mathtt{j}_{P_{q-1}} \circ \partial(\gamma_i)$$
$$\text{with} \quad H^{\mathtt{g}}_{q-1} = (\mathtt{j}_{P^{\mathtt{g}}_{q-1}} \circ \partial \circ \mathtt{i}_{S^{\mathtt{g}}_q})^{-1} \qquad \text{and} \quad H_{q-1} = (\mathtt{j}_{P_{q-1}} \circ \partial \circ \mathtt{i}_{S_q})^{-1}$$

As previously explained, we have $P_{q-1} = P^{\mathtt{g}}_{q-1}$ and $S_q = S^{\mathtt{g}}_q$. Thus, we obtain $H^{\mathtt{g}}_{q-1} \circ$

$\mathtt{j}_{P^{\mathtt{g}}_{q-1}} = H_{q-1} \circ \mathtt{j}_{P_{q-1}}$, which yields $g(\gamma_i) = g^{\mathtt{g}}(\gamma_i)$. Similarly, we obtain $f^*(\gamma_i) = f^{\mathtt{f}^*}(\gamma_i)$. Hence, $g(\gamma_i) = \mathtt{g}_i$ and $f^*(\gamma_i) = \mathtt{f}_i$.

As a result, $\{g(\gamma_i)\}_{i\leq\beta_q}$ is a q-homology basis of (K,∂), so it is a set of cycles such that $\left\{[g(\gamma_i)]^K\right\}_{i\leq\beta_q}$ is linearly independent in $H_q(K)$. Thus, we can use Lemma 1.6.13 to obtain a perfect HDVF $X' = (P', S', C')$ (with reduction (f', g', h')) such that $X \prec X'$ and $\{\gamma_i\}_{i\leq\beta_q} \subseteq C'_q$. By cardinality we obtain $\{\gamma_i\}_{i\leq\beta_q} = C'_q$.

To conclude, $\mathtt{g}_i = g(\gamma_i)$ is a cycle in (K,∂) so we use Lemma 1.6.15 on $X \prec X'$ to obtain $g'(\gamma_i) = g(\gamma_i) = \mathtt{g}_i$. As a result, X' induces the homology basis $\{\mathtt{g}_i\}_{i\leq\beta_q}$. With dual arguments, $f'^*(\gamma_i) = f^*(\gamma_i) = \mathtt{f}_i$, so X' induces $\{\mathtt{f}_i\}_{i\leq\beta_q}$ as well. □

Hence, homology and cohomology bases simultaneously computed via HDVFs follow strict combinatorial rules. This yields, for instance, that computation of shortest homology and cohomology bases cannot be carried out simultaneously. A complement of this result, with a higher point of view, is presented in Section 4.3.3 (see Proposition 4.3.4).

2.3.5. Characterization of homology bases computed by other methods

In this section, we discuss how the characterization of Section 2.3 transfers to bases computed by other methods.

First, it is important to distinguish computational homology approaches (such as HDVFs, but also Smith Normal Form, persistent homology and tri-partitions) and heuristics for computing small homology generators (the question of computing minimal ones is NP-hard, see for instance T. Dey et al. 2020). Methods related to this later question usually combine former approaches together with solvers and heuristics to minimize generators, hence homology bases may not be explicit (see for instance Figure 2.4, K_3). Now, the remainder of this section examines the links between HDVFs and other approaches and deduces properties on the bases they compute.

Discrete Morse theory

As stated in Section 1.6.4, HDVFs generalize Discrete Morse Theory, in particular a *discrete gradient vector field* is a HDVF. Therefore, Theorem 2.3.4 naturally applies, and Discrete Morse Theory computes only explicit bases.

Smith Normal Form (SNF)

As stated and detailed in Section 1.6.4, standard elimination steps of SNF can be encoded via HDVFs (through `A` operations). Therefore, Theorem 2.3.4 applies to methods that deduce homology bases from the SNF of the boundary matrix (when

homology is computed over a field). Such methods include PELTIER et al. 2006, that uses a modified version of the SNF of the boundary matrix.

Tri-partitions

In Section 2.2, Propositions 2.2.4 and 2.2.5 underlines the fact that perfect HDVFs and tri-partitions represent the same object. In particular, their induced homology and cohomology bases coincide. Thus, the characterization results of Section 2.3 extend to tri-parttions:

Theorem 2.3.11. *A q-homology (resp. q-cohomology) basis is explicit if and only if it is induced by a q-tri-partition.*

Graphs can be considered as special cases of complexes of maximal dimension 1. In such context, the homology bases obtained using 1-tri-partitions (i.e. using spanning trees on the graph) are called *strictly fundamental cycle bases*, see Definition 15 of LIEBCHEN et al. 2007. Thus, for graphs, a homology basis is explicit if and only if it is a strictly fundamental cycle basis.

With this observation, explicit homology bases can be understood as the dimensional generalization of strictly fundamental cycle bases.

This observation also allows answering the following question: are minimal explicit bases easier to compute than standard minimal homology bases? Unfortunately, the answer is negative, indeed, it was proved in DEO et al. 1982 that this problem is also NP-hard for strictly fundamental cycle bases in graphs.

Persistent homology

The standard persistent homology algorithm Algorithm 1 of EDELSBRUNNER et al. 2002 can be used to compute homology bases at each step of a filtration. More precisely, each positive cell τ is associated with a unique cycle consisting of τ together with negative cells. In the following chapter, we show that this computation can be emulated by a HDVF completion. Precisely, we prove that the HDVF-based algorithm for persistent homology, introduced in GONZALEZ-LORENZO, BAC, MARI, et al. 2017, reproduces exactly the steps of the standard persistent homology algorithm. As a consequence, Theorem 2.3.4 implies that these persistent homology bases are explicit, see Section 3.2.

Note that other works such as PHAM et al. 2024; ČUFAR et al. 2023 deduce persistent homology bases from boundary matrix reductions. Again, as boundary matrix reductions can be reproduced by a HDVF completion, the persistent homology bases computed by these methods are explicit.

2.4. Conclusion

This chapter refines the understanding of the combinatorial structure of HDVFs. We first establish a close relationship between perfect HDVFs and *tri-partitions*, a

combinatorial structure introduced in H. EDELSBRUNNER and ÖLSBÖCK 2020. Tri-partitions can be considered as "dimensional layers" of perfect HDVF. Conversely, a perfect HDVF can be viewed as a "stack" of tri-partitions. This connection may lead to the combinatorial investigation of harmonic cycles and discrete Helmholtz-Hodge decomposition, which are derived from the combinatorial Laplacian $\Delta = \partial\partial^* + \partial^*\partial$ (see for instance GOLDBERG 2002).

This chapter also introduces a characterization of the homology bases computed by HDVFs and related methods in computational homology. We introduce *explicit* homology bases, which can be viewed as a generalization of strictly fundamental bases for graphs to arbitrary combinatorial chain complexes. We prove that being an explicit homology basis is equivalent to being induced by an HDVF. We then establish a similar result for cohomology bases and provide a necessary and sufficient condition under which a pair of explicit homology and cohomology bases can be computed by a single HDVF. Finally, we show how these results apply to other approaches, including tri-partitions, discrete Morse theory, boundary matrix reduction methods, and standard persistent homology algorithms.

These results suggest that the combinatorial structure of homology computations may play an important role in practical applications of homology-based methods. For instance, these results may give some insights on methods based on homology bases, such as the hole closing and opening operations introduced in H. EDELSBRUNNER and ÖLSBÖCK 2019. They also lead to a better understanding of the isomorphisms constructed in GONZALEZ-LORENZO, BAC, and GAZULL 2024, as discussed in Chapter 4.

Moreover, explicit bases and HDVFs are inherently combinatorial objects, integrating combinatorics into the algebraic framework of homology. This observation raises the broader question of whether such objects can reveal additional information about the topology of the underlying complex. This question leads to the notion of *homology configurations*, which is presented in Chapter 4.

3. The Space of HDVFs and Homology Computations

Sommaire

3.1. Introduction 86
3.2. Persistent Homology and HDVFs 87
3.2.1. The standard abstract persistence algorithm 87
3.2.2. Emulating persistent homology with HDVF 88
3.2.3. Persistent generators 93
3.2.4. About the filtration space 94
3.3. The Graph of HDVFs 97
3.3.1. The graph of perfect HDVFs 97
3.3.2. The connectedness algorithm 98
3.3.3. Proof of the connectedness 100
3.3.4. Experimental results on the graph of HDVFs 104
3.4. Adapting a Perfect HDVF to a Given Explicit Basis 107
3.4.1. The adaptation algorithm 108
3.4.2. Proof of the correctness of the adaptation 110
3.5. Conclusion 113

3.1. Introduction

As mentioned previously, HDVFs provide a representation of the homology computations performed by discrete Morse theory, standard persistent homology and related approaches. In this chapter, we investigate the structure of these homology computations. Firstly, we study persistent homology through the lens of HDVF, and discuss the space of filtrations and their associated persistent computations. In particular, we show how persistent homology bases can be extracted from a final perfect HDVF, and we we introduce a hierarchy of viewpoints on the space of filtrations.

Secondly, we study the space of HDVFs itself, by equipping the set of perfect HDVFs with a graph structure. Our main result is that, for a fixed complex, all perfect HDVFs are connected through elementary swapping operations. In addition, we present related experimental results and provide an algorithm to adapt a perfect HDVF to a given explicit basis. These results about the graph of HDVFs can be found in a preprint, see GAZULL, GONZALEZ-LORENZO, et al. 2025b.

Throughout this chapter, we consider combinatorial chain complexes, which we simply refer to as *complexes*. We consider homology over a fixed field $\mathscr{R}$.

3.2. Persistent Homology and HDVFs

The work of GONZALEZ-LORENZO, BAC, MARI, et al. 2017 shows that the HDVF computation algorithm can be modified to compute persistent homology. The modification consists in introducing primary-secondary pairs according to the input filtration. The proof relies on the observation that both HDVF computations and persistent homology computations correspond to pseudo-diagonalizations of the boundary matrix. A similar result is obtained in GONZALEZ-DIAZ, ION, et al. 2011, where the authors adapt the computation of *AT-models* (i.e., perfect reductions) to emulate persistent homology. Likewise, H. EDELSBRUNNER and ÖLSBÖCK 2020 shows that every monotonic ordering of cells induces a unique tri-partition. However, the proofs of these results heavily rely on investigating matrix reduction algorithms, leaving behind interesting combinatorial details of this computation.

This section refines the equivalence between HDVF and persistent homology computations, with a more combinatorial point of view. To do so, we prove that the HDVF algorithm for persistence is a direct translation of the standard abstract persistent algorithm EDELSBRUNNER et al. 2002 into the HDVF framework. This perspective allows us to define persistent homology generators and bases (see Section 3.2.3). In particular, we show that they can be read directly in the final perfect HDVF computed by this algorithm.

For clarity, given a complex, we define a *filtration* as a function from its cells to $\mathbb{R}$, such that every cell has a higher value than its boundary cells. Every filtration naturally induces an ordering of the cells, that may not be strict as several cells may appear at the same time. We define a *filter* as a strict ordering of the cells, such that every cell arrives later than its boundary cells. In the following algorithms, we only consider filters.

3.2.1. The standard abstract persistence algorithm

We first present the standard abstract persistent homology algorithm, introduced by EDELSBRUNNER et al. 2002. We consider a filter $(\tau_i)_{i=1}^N$ and its associated sequence of combinatorial chain complexes $\emptyset = K^0 \subseteq \cdots \subseteq K^N$, introducing the cell τ_i at step i. As every cell arrives after the cells of its boundary, every (K^i, ∂^i) is a subcomplex of (K^j, ∂^j) (with $i < j$) closed under the boundary map ∂^j.

The algorithm is presented in Algorithm 1 and uses the following definition:

Definition 3.2.1 (positive and negative cells EDELSBRUNNER et al. 2002)**.** In Algorithm 1, a cell τ_i is *positive* if it is contained in a cycle of K^i. It is negative otherwise. A positive cell creates a homology class whereas a negative cell deletes a homology class.

Algorithme 1 : Abstract persistence algorithm EDELSBRUNNER et al. 2002

Data : A complex (K, ∂) and a filter $(\tau_i)_{i=1}^N$
Result : L: the persistence intervals
$L \leftarrow \emptyset$;
for $i = 1$ *to* N **do**
  if τ_i *is a negative cell* **then**
    $k \leftarrow$ highest index of the positive cells associated to $\partial(\tau_i)$ $(\star)$;
    $L_{\dim(\tau_i)} \leftarrow L_{\dim(\tau_i)} \cup \{(k, i)\}$ (add birth-death interval);
for *unpaired positive* τ_i **do**
  $L_{\dim(\tau_i)} \leftarrow L_{\dim(\tau_i)} \cup \{(i, +\infty)\}$ (add the remaining birth-death intervals);

In Algorithm 1, the cycle associated to a positive cell τ_i is called ζ_i and is the one and only cycle composed of τ_i and negative cells τ_j with $j < i$. Therefore, a positive q-cell τ_j is associated to a unique homology class $[\zeta_j]_q^{K^i} \in H_q(K^i)$.

The step $(\star)$ of the algorithm works as follows: as τ_i is a negative cell, $\partial(\tau_i)$ is a cycle that is not a boundary in K^{i-1}. Hence, there exists coefficients $(\lambda_j)_{j\in J}$ such that $[\partial(\tau_i)]^{K^{i-1}} = \sum_{j\in J} \lambda_j [\zeta_j]^{K^{i-1}}$, where $J \subseteq \{1, \dots, i-1\}$ denotes the indices of the positive cells of K^{i-1}. k is then defined as the max of $\{j \mid \lambda_j \neq 0\}$, i.e. it represents the youngest non-paired positive cell τ_k such that $\left\langle [\partial(\tau_i)]^{K^{i-1}}, [\zeta_k]^{K^{i-1}} \right\rangle \neq 0$.

3.2.2. Emulating persistent homology with HDVF

Now, we recall the algorithm of GONZALEZ-LORENZO, BAC, MARI, et al. 2017, which computes the persistent homology of a filter by constructing a perfect HDVF (see Algorithm 2). We then prove in Propositions 3.2.3 and 3.2.5 that each step of Algorithm 2 is equivalent to the corresponding step of Algorithm 1. The complexity of both algorithms is the same as that of computing a perfect HDVF, namely at most cubic in the number of cells.

Algorithme 2 : HDVF persistence algorithm

Data : A complex (K, ∂) and a filter $(\tau_i)_{i=1}^N$
Result : L: the persistence intervals
$L \leftarrow \emptyset$; $(P, S, C) \leftarrow (\emptyset, \emptyset, \overline{K})$; $d \leftarrow \partial$;
for $i = 1$ *to* N **do**
  if $d(\tau_i) \neq 0$ **then**
    $k \leftarrow max\{j \;/\; \langle d(\tau_i), \tau_j \rangle \neq 0\}$ (find the youngest appropriate critical cell) $(\star)$;
    $(P, S, C) \leftarrow (P \cup \{\tau_k\},\ S \cup \{\tau_i\},\ C \backslash \{\tau_i, \tau_k\})$ (and update d);
    $L_{dim(\tau_i)} \leftarrow L_{dim(\tau_i)} \cup \{(k, i)\}$ (add birth-death interval);
for τ_i *in* C **do**
  $L_{dim(\tau_i)} \leftarrow L_{dim(\tau_i)} \cup \{(i, +\infty)\}$ (add the remaining birth-death intervals);

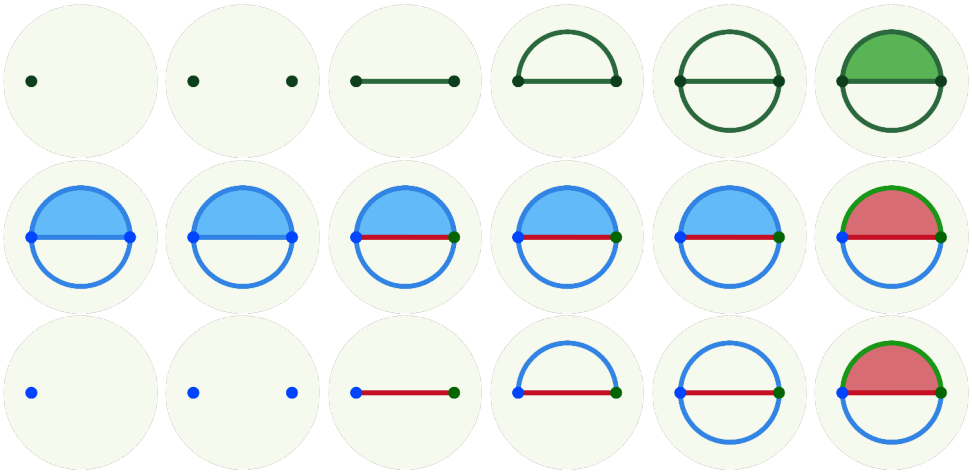


Figure 3.1 – Example of an execution of Algorithm 2 with the sequence of HDVFs. First row represents the filtration. Second row represents the HDVFs X^i. Third row represents the perfect HDVFs $\check{X}^i$, see Lemma 3.2.1.

Table 3.1 – A parallel between the elements of both algorithms.

Standard Persistence Algorithm 1	HDVF Persistence Algorithm 2
positive cell (creates a hole)	primary or critical cell ($d^{i-1}(\tau_i) = 0$)
negative cell (fills a hole)	secondary cell ($d^{i-1}(\tau_i) \neq 0$)
persistent q-homology generator ζ_i	$g^i_q(\tau_i) = z_q(\tau_i)$
criterion for step $(*)$: $\langle [\partial(\tau_i)]^{K^{i-1}}, [\zeta_j]^{K^{i-1}} \rangle \neq 0$	$\langle d^{i-1}(\tau_i), \tau_j \rangle \neq 0$

Let $(\tau_i)_{i=1}^N$ be a filter, and let $\emptyset = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^N$ be the sequence of nested complexes derived from this filter. Let $(X^i)_{i=0}^N$ be the sequence of HDVF built during the `for` loop of Algorithm 2. In particular we have $X^0 \prec \cdots \prec X^N$. Each $X^i = (P^i, S^i, C^i)$ is a HDVF (not necessarily perfect) for (K, ∂) and it induces a reduction (f^i, g^i, h^i) to $(\operatorname{Span} C^i, d^i)$. We denote by $X := X^N$ the final perfect HDVF obtained at the end of the filter, with canonical (co)cycle maps z_q and z^q. See Figure 3.1 for an execution of this algorithm. A comparison between the concepts used in both algorithms is summarized in Table 3.1.

We begin by showing that the algorithm produces a perfect HDVF at each step of the filtration. We denote by $\check{X}^i = (P^i \cap \overline{K}^i, S^i \cap \overline{K}^i, C^i \cap \overline{K}^i)$ the restriction of X^i to the subcomplex (K^i, ∂^i).

Lemma 3.2.1. *$\check{X}^i$ is a perfect HDVF for (K^i, ∂^i).*

Proof. By induction:
Base case: $\check{X}^0 = (\emptyset, \emptyset, \overline{K} \cap \overline{K}^0) = (\emptyset, \emptyset, \emptyset)$ is a perfect HDVF for $(\emptyset, \partial^0)$.
Induction step: suppose that $\check{X}^{i-1}$ is a perfect HDVF for $(K^{i-1}, \partial^{i-1})$. We show that $\check{X}^i$ is a perfect HDVF for (K^i, ∂^i). We first prove that $\check{X}^i$ is a HDVF such that $\check{X}^{i-1} \prec \check{X}^i$. There are two cases:

- $d^{i-1}(\tau_i) = 0$:
  in this case, we have $X^{i-1} = X^i$ so $\check{X}^i = (\check{P}^{i-1}, \check{S}^{i-1}, \check{C}^{i-1} \cup \{\tau_i\})$. $\check{X}^i$ is a HDVF because $\mathtt{j}_{\check{P}^{i-1}} \partial \mathtt{i}_{\check{S}^{i-1}}$ is invertible by induction.

  We now prove that $\check{X}^i$ is perfect, i.e. that $\check{d}^i = 0$. Let $\gamma \in \check{C}^{i-1} \cup \{\tau_i\}$.

  As $X^{i-1} = X^i$, we have $d^{i-1} = d^i$, therefore $\check{d}^i(\gamma) = \mathtt{j}_{K^i} d^i(\gamma) = \mathtt{j}_{K^i} d^{i-1}(\gamma)$.
  Either $\gamma = \tau_i$, then $d^{i-1}(\tau_i) = 0$ by assumption, so $\check{d}^i(\gamma) = 0$.
  Otherwise, $\gamma \in \check{C}^{i-1}$. In this case, we consider the chain $\check{g}^{i-1}(\gamma)$ and we have:

$$\begin{aligned} \partial^i \check{g}^{i-1}(\gamma) &= \partial^{i-1} \check{g}^{i-1}(\gamma) && \text{as } (K^{i-1}, \partial^{i-1}) \text{ is closed under } \partial^i \text{ in } (K^i, \partial^i) \\ &= 0 && \text{as } \check{X}^{i-1} \text{ is perfect (by induction) in } (K^{i-1}, \partial^{i-1}) \end{aligned}$$

  Therefore, $\check{g}^{i-1}(\gamma)$ is a cycle of K^i, composed of γ plus cells in $\check{S}^{i-1} \subseteq \check{S}^i$. By Lemma 1.6.3, $\check{g}^{i-1}(\gamma) = \check{g}^i(\gamma)$. Hence, we get

$$\check{g}^i d^i(\gamma) = \partial^i \check{g}^i(\gamma) = 0$$

  Which gives $d^i(\gamma) = 0$ as $\check{g}^i$ is injective.

- $d^{i-1}(\tau_i) \neq 0$:
  then there exists a cell $\tau_k \in C^{i-1} \cap \overline{K}^{i-1} = \check{C}^{i-1}$ such that $X^i = \mathtt{A}^{\tau_i}_{\tau_k} X^{i-1}$. Hence, we have $\check{X}^i = (\check{P}^{i-1} \sqcup \{\tau_k\}, \check{S}^{i-1} \sqcup \{\tau_i\}, \check{C}^{i-1} \setminus \{\tau_i, \tau_k\})$ so $\check{X}^i$ is a completion of $\check{X}^{i-1}$.

  X^i is a HDVF by construction, so $\check{X}^i$ is also a HDVF because $\mathtt{j}_{\check{P}^i} \partial \mathtt{i}_{\check{S}^i} = \mathtt{j}_{P^i} \partial \mathtt{i}_{S^i}$.

  We now prove that $\check{X}^i$ is perfect, i.e. that $\check{d}^i = 0$. Let $\gamma \in \check{C}^i \subseteq \check{C}^{i-1}$.

  As in the previous case, we have:

$$\begin{aligned} \partial^i \check{g}^{i-1}(\gamma) &= \partial^{i-1} \check{g}^{i-1}(\gamma) && \text{as } (K^{i-1}, \partial^{i-1}) \text{ is closed under } \partial^i \text{ in } (K^i, \partial^i) \\ &= 0 && \text{as } \check{X}^{i-1} \text{ is perfect (by induction) in } (K^{i-1}, \partial^{i-1}) \end{aligned}$$

  Therefore, $\check{g}^{i-1}(\gamma)$ is a cycle of K^i, composed of γ plus cells in $\check{S}^{i-1} \subseteq \check{S}^i$. By Lemma 1.6.3, $\check{g}^{i-1}(\gamma) = \check{g}^i(\gamma)$. Hence, we get

$$\check{g}^i d^i(\gamma) = \partial^i \check{g}^i(\gamma) = 0$$

  Which gives $d^i(\gamma) = 0$ as $\check{g}^i$ is injective. □

As a result, in addition to the sequence $X^0 \prec \cdots \prec X^N$ of HDVFs for (K, ∂), we obtain a sequence of perfect HDVFs $\check{X}^0 \prec \cdots \prec \check{X}^N$, whose reductions are displayed in this diagram:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_{q+1}} & K_q & \xrightarrow{\partial_q} & \cdots & \qquad & \cdots \xrightarrow{d^i_{q+1}} \operatorname{Span} C^i_q \xrightarrow{d^i_q} \cdots \\ & \xleftarrow[h^i_q]{} & \uparrow g^i_q \downarrow f^i_q & \xleftarrow[h^i_{q-1}]{} & & & \end{array}$$

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^i_{q+1}} & K^i_q & \xrightarrow{\partial^i_q} & \cdots & \qquad & \cdots \xrightarrow{0} \operatorname{Span} \check{C}^i_q \xrightarrow{0} \cdots \\ & \xleftarrow[h^i_q]{} & \uparrow \check{g}^i_q \downarrow \check{f}^i_q & \xleftarrow[h^i_{q-1}]{} & & & \end{array}$$

Moreover, we obtain the following corollary.

Corollary 3.2.2 (preservation of the canonical cycle)**.** *Denote $z_q^i := id - h^i\partial$, the canonical **cycle** operator. If $x \in K_q^i$, then $z_q^i(x) = z_q^j(x)$ for all $j \geq i$. In particular, if $x \in K^i \cap \operatorname{Span} C^i \cap \operatorname{Span} C^j$, $g^i(x) = g^j(x)$.*

Proof. As every (K^i, ∂^i) is a subcomplex of (K, ∂) closed under ∂, we have $\ker \partial^i \subseteq \ker \partial$. As a result, Lemma 3.2.1 and Lemma 1.6.3 imply that $\{z_q^i(x)\} = \ker \partial \cap (x + \operatorname{Span} S^i)$ and $\{z_q^j(x)\} = \ker \partial \cap (x + \operatorname{Span} S^j)$. As $S^i \subseteq S^j$, we have $z_q^i(x) \in \ker \partial \cap (x + \operatorname{Span} S^j)$ so $z_q^i(x) = z_q^j(x)$. □

This result shows that canonical cycles remain unchanged along the filtration. As a consequence, each homology class appearing in the filtration is associated with a unique cell τ_i and a unique cycle $z_q^i(\tau_i)$. This point is discussed and illustrated in Section 3.2.3.

Remark. The canonical cocycles are not necessarily preserved during the filtration. This is due to the fact that the complexes (K^i, ∂^i) are not closed under the coboundary operator ∂^*. By dualizing the whole process, it is possible to define persistent cohomology, over "cofiltrations". In this case, canonical cocycles are preserved, but not canonical cycles. See SILVA et al. 2011 for more information about dualities in persistent homology.

We now prove that the condition of the `if` in both algorithms coincide.

Proposition 3.2.3. *In Algorithm 2: a cell τ_i is positive if and only if $d^{i-1}(\tau_i) = 0$. Therefore, the conditions inside the loops of Algorithm 1 and Algorithm 2 are equivalent.*

Proof.

- $\Longrightarrow$. If τ_i is positive, then there exists $x = \tau_i + y$ a cycle in (K^i, ∂), with $y \in K^{i-1}$.

  x is not a boundary in K^i, otherwise τ_i would be in the boundary of a cell that arrived before in the filtration. Hence, its associated homology class is not trivial. As X^i restricted to K^i is perfect (by Lemma 3.2.1), there exists $c \in \operatorname{Span} C^i \cap K^i$ and $x' \in K^i$ such that $x = g^i(c) + \partial(x')$.

  Hence, we have $g^i(c) = \tau_i + y - \partial^i(x')$. $\partial^i(x')$ is in K^{i-1} (because x' is in K^i and so all its boundary cells are in K^{i-1}), therefore, we have that $g^i(c) \in \tau_i + K^{i-1}$.

  By contradiction, suppose that $d^{i-1}(\tau_i) \neq 0$. Then, Algorithm 2 will set τ_i as a secondary cell in X^i. This implies that $C_q^i = C_q^{i-1}$ (where q is the dimension of τ_i), so $c \in \operatorname{Span} C^{i-1} \cap K^i = \operatorname{Span} C^i \cap \operatorname{Span} C^{i-1} \cap K^{i-1}$ (as $\tau_i \notin C^{i-1}$). By Corollary 3.2.2 we get $g^i(c) = g^{i-1}(c)$, which implies $g^{i-1}(c) \in \tau_i + K^{i-1}$, which contradicts the fact that $g^{i-1}(c) \in K^{i-1}$.

  As a result, $d^{i-1}(\tau_i) = 0$.

- $\Longleftarrow$. If $d^{i-1}(\tau_i) = 0$, then $\partial g^{i-1}(\tau_i) = g^{i-1}d^{i-1}(\tau_i) = 0$ so $g^{i-1}(\tau_i)$ is a cycle. Moreover, $g^{i-1}(\tau_i)$ belongs to (K^i, ∂) as it is made of τ_i and secondary cells of S^{i-1}, which is inside (K^i, ∂^i).

  Therefore, τ_i belongs to a cycle in (K^i, ∂^i).

□

As a result, we obtain a direct correspondence between negative and positive cells with secondary, primary and critical cells.

As a result, Algorithm 2 and Proposition 3.2.3 implies the following corollary, which states a direct connection between negative and positive cells with secondary and primary/critical cells.

Corollary 3.2.4.

$$\begin{aligned}\tau_i \text{ is negative} &\iff d^{i-1}(\tau_i) \neq 0 \iff \tau_i \in S^i \iff \tau_i \in S^N\\ \tau_i \text{ is positive} &\iff d^{i-1}(\tau_i) = 0 \iff \tau_i \in C^i \iff \tau_i \in C^N \sqcup P^N\end{aligned}$$

Finally, it remains to prove that the steps ($\star$) in both algorithms are equivalent. Precisely, we must show that being a positive cell $\tau_j \in K^{i-1}$ associated to $\partial(\tau_i)$ is equivalent to $\langle d^{i-1}(\tau_i), \tau_j\rangle \neq 0$. This is stated formally in Proposition 3.2.5. Recall that, for a positive cell τ_i, ζ_i is the unique cycle composed of τ_i and negative cells τ_j with $j < i$ (see Algorithm 1).

Proposition 3.2.5. *If τ_i is negative and τ_j is a non-paired positive cell (with $j < i$), then:*

$$\left\langle [\partial(\tau_i)]^{K^{i-1}}, [\zeta_j]^{K^{i-1}} \right\rangle \neq 0 \Longleftrightarrow \langle d^{i-1}(\tau_i), \tau_j\rangle \neq 0$$

Hence, the steps ($\star$) in Algorithms 1 and 2 are equivalent.

To prove this proposition, we first need the following lemma.

Lemma 3.2.6. *If τ_i is a negative cell:*

$$[\partial(\tau_i)]^{K^{i-1}} = [g^{i-1}(c)]^{K^{i-1}} \Longleftrightarrow d^{i-1}(\tau_i) = c$$

Proof.

- $\Longrightarrow$. There exist $x \in K^{i-1}$ such that $\partial(\tau_i) = g^{i-1}(c) + \partial(x)$. By Lemma 3.2.1, $\check{X}^{i-1}$ is perfect. We denote by $(\check{f}, \check{g}, \check{h})$ its associated perfect reduction over $(K^{i-1}, \partial^{i-1})$. Then, by applying f^{i-1} to $\partial(\tau_i) = g^{i-1}(c) + \partial(x)$ we get:

  $$\begin{aligned}&d^{i-1}f^{i-1}(\tau_i) = id(c) + f^{i-1}\partial(x) && \text{because } f^{i-1}\partial = d^{i-1}f^{i-1} \text{ and } f^{i-1}g^{i-1} = id\\ &d^{i-1}f^{i-1}(\tau_i) = c + \check{f}\partial^{i-1}(x) && \text{as } x \in K^{i-1},\ \partial(x) = \partial^{i-1}(x)\\ &d^{i-1}f^{i-1}(\tau_i) = c && \text{as } \check{f}\partial^{i-1} = 0 \text{ because the reduction is perfect}\end{aligned}$$

  Finally, as $\tau_i \in C^{i-1}$, we have $f^{i-1}(\tau_i) = \tau_i$. This implies $d^{i-1}(\tau_i) = c$.

- $\Longleftarrow$. First, we have $g^{i-1}(c) = g^{i-1}d^{i-1}(\tau_i) = \partial g^{i-1}(\tau_i)$ because of standard reduction properties. In addition, $g^{i-1}(\tau_i) = \tau_i + s$ with $s \in S^{i-1} \subseteq K^{i-1}$. Hence, $\partial g^{i-1}(\tau_i) = \partial(\tau_i) + \partial(s)$, with $\partial(s)$ a boundary of K^{i-1}. As a result: $[g^{i-1}(c)]^{K^{i-1}} = [\partial(\tau_i)]^{K^{i-1}} + [\partial(s)]^{K^{i-1}} = [\partial(\tau_i)]^{K^{i-1}}$. □

Proof. [proof of Proposition 3.2.5] Firstly, we have $g_j(\tau_j) = \zeta_j$ by Lemma 1.6.3. Indeed, ζ_j is a cycle equal to τ_j plus a sum of negative cells, and negative cells are all secondary cells in X^j. τ_j was not paired so we also have $g^{i-1}(\tau_j) = g^j(\tau_j)$.

$\partial(\tau_i)$ is a cycle that is not a boundary in K^{i-1}. Hence, there exists coefficients $(\lambda_j)_{j\in J}$ such that $[\partial(\tau_i)]^{K^{i-1}} = \sum_{j\in J} \lambda_j [\zeta_j]^{K^{i-1}}$. $J \subseteq \{1,\dots,i-1\}$ denotes the indices of the positive cells of K^{i-1} (that are not paired).

Therefore, we obtain

$$[\partial(\tau_i)]^{K^{i-1}} = \sum_{j\in J} \lambda_j [g^{i-1}(\tau_j)]^{K^{i-1}} = [g^{i-1}(\sum_{j\in J} \lambda_j \tau_j)]^{K^{i-1}}$$

By Lemma 3.2.6, with $c = \sum_{j\in J} \lambda_j \tau_j$, we obtain $[\partial(\tau_i)]^{K^{i-1}} = \sum_{j\in J} \lambda_j [\zeta_j]^{K^{i-1}} \Longleftrightarrow d^{i-1}(\tau_i) = \sum_{j\in J} \lambda \tau_j$. This implies the proposition, by replacing the λ_j with the corresponding $\langle\cdot,\cdot\rangle$. □

Combining Propositions 3.2.3 and 3.2.5, we conclude that Algorithm 1 and Algorithm 2 are equivalent for computing the persistence intervals associated to a filter.

The proof of this equivalence also revealed properties that can be read in the successive HDVFs, such as persistent generators and bases, which are discuss in the next subsection. Another example is that given τ_i a negative cell, $g^{i-1}(\tau_i)$ is a chain that represents how the hole is filled at step i.

3.2.3. Persistent generators

Algorithm 2 not only computes persistence intervals, but also HDVFs and reductions for each step of the filter. In particular, it computes a q-homology basis for each (K^i, ∂), called the *i-th persistent q-homology basis* and whose elements are the *persistent q-homology generators*.

Formally, let (f^i, g^i, h^i) be the reduction associated with X^i. Then, the i-th persistent q-homology basis is given by $\{g_q^i(\gamma)\}_{\gamma\in C_q^i}$. By Theorem 2.3.4, these persistent homology bases are explicit. Moreover, Corollary 3.2.2 implies that persistent generators are preserved throughout the lifetime of the homology classes they represent. Equivalently, for all $\gamma \in C_q^i$, we have $z_q(\gamma) = g_q^i(\gamma)$, where z_q denotes the canonical cycle map of the final HDVF X^N. Therefore, $\{g_q^i(\gamma)\}_{\gamma\in C_q^i}$ defines a *persistent basis* in the sense of T. K. Dey, Hou, et al. 2019; Wu et al. 2017; Obayashi 2017.

As a consequence, each point (i, j) in the persistence diagram corresponds to a unique cell τ_i and a unique persistent homology generator $z_q(\tau_i)$. More precisely, τ_i is a critical cell from step i up to step j, and becomes a primary cell after step j, unless

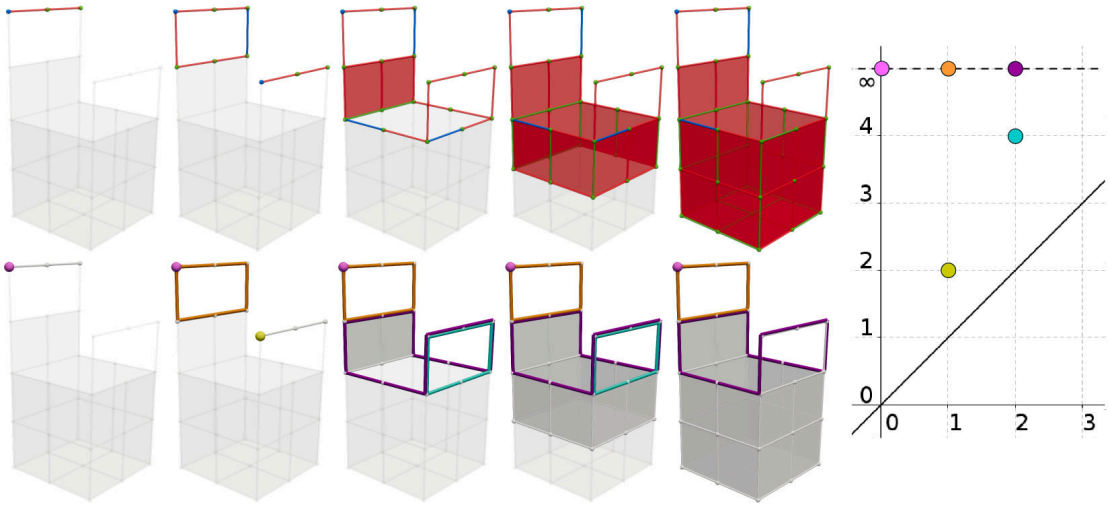


Figure 3.2 – Example of persistent homology computed by HDVF. First row: the filtration $K_0 \subseteq \cdots \subseteq K_4$ and its HDVFs $X^0, \ldots, X^4$. Second row: the ith-persistent homology bases for $i = 0, \ldots, 4$ (dimensions 0 and 1 are represented). On the right: the persistence diagram of the filtration, each point (i, j) corresponds to a homology class whose lifetime is $[i, j]$, its color matches the color of the corresponding persistent generator.

$j = N$, in which case τ_i remains critical in the final HDVF. In particular, persistent homology generators can be read directly in the final perfect HDVF via the canonical cycle map. An illustration of persistent homology bases computed using HDVFs is given in Figure 3.2.

3.2.4. About the filtration space

A filter induces a unique perfect HDVF through Algorithm 2. A natural question is whether every perfect HDVF arises from some filter. The answer is positive, as shown in Proposition 3.2.7.

We consider a filter $(\tau_i)_{i=1}^N$ and denote by $X^i = (P^i, S^i, C^i)$ the corresponding sequence of HDVFs, with X^N its final HDVF.

Proposition 3.2.7. *Let $X = (P, S, C)$ be a perfect HDVF. Suppose that the filter $(\tau_i)_{i=1}^N$ satisfies the following property for every dimension q:*

$$(\tau_i, \tau_j, \tau_k) \in S_q \times C_q \times P_q \implies i < j < k$$

Alternatively: cells of S_q arrive before cells of C_q, which arrive before cells of P_q. Then we have that $X = X^N$.

Proof.

- First, we show that if τ_i is positive, then τ_i is not in S_q.
  τ_i is positive, so there exists a q-cycle x of K^i such that $\langle x, \tau_i \rangle \neq 0$. By contra-

diction, suppose that $\tau_i \in S_q$. As cells of S_q arrive before cells of $C_q \sqcup P_q$, it means that the q-cells of K^i are all in S_q. As a result, $x \in \ker\partial \cap \operatorname{Span} S_q$ so $x = 0$ (by Lemma 1.6.3), which contradicts $\langle x, \tau_i \rangle \neq 0$.

- Conversely, we show that if τ_i is in $C_q \sqcup P_q$, then τ_i is positive.
  Consider $x := z_q(\tau_i) \in (\tau_i + \operatorname{Span} S_q) \cap \ker\partial$. As cells of S_q arrive before cells of $C_q \sqcup P_q$ and $\tau_i \in C_q \sqcup P_q$, we have that $\operatorname{Span} S_q \subseteq K^i$. Hence, x is a cycle of K^i, which is non-zero as $\tau_i \neq S$. As a result, τ_i is contained in a cycle of K^i so it is positive (see Definition 3.2.1).

All in all we obtain that τ_i is positive if and only if $\tau_i \in C \sqcup P$. Using Corollary 3.2.4 we obtain that $S^N = S$ and $C^N \sqcup P^N = C \sqcup P$. It remains to prove that secondary cells are only paired with cells of P and not of C.

Suppose we have $c := d^{i-1}(\tau_i) \neq 0$. By contradiction, suppose that c does not have any cell of P. In this case, it means $c \in \operatorname{Span} C$, as $c \in \operatorname{Span} C^{i-1} \subseteq \operatorname{Span}(C \sqcup P)$ in general. We define $x := g^{i-1}(d^{i-1}(\tau_i)) = \partial g^{i-1}(\tau_i)$. Hence, x is a boundary, so it is a cycle: $x \in \ker\partial \cap (c + \operatorname{Span} S^{i-1})$. As $S^{i-1} \subseteq S$, we obtain that $x \in \ker\partial \cap (c + \operatorname{Span} S)$ so $x = g(c)$ by Lemma 1.6.3. Therefore, $g(c)$ is boundary, which implies that $c = 0$ (as $(g(\gamma))_{\gamma \in C_q}$ is a homology basis), which contradicts $c := d^{i-1}(\tau_i) \neq 0$.

As a result, there exists a cell $\tau_j \in P$ such that $\langle d^{i-1}(\tau_i), \tau_j \rangle \neq 0$. Because cells of C_q arrive before cells of P_q, the step in Algorithm 2 where we perform "$k \leftarrow max\{j \,/\, \langle d^{i-1}(\tau_i), \tau_j \rangle \neq 0\}$" always results in choosing k such that $\tau_k \in P$.

Hence, every cell of C is never paired, so we obtain $C \subseteq C^N$. By cardinality ($|C_q| = \beta_q = |C_q^N|$), we obtain that $C = C^N$. As HDVF are partitions, we obtain that $(P, S, C) = (P^N, S^N, C^N)$. □

An immediate consequence of this result is that some persistent homology computations cannot be captured by discrete Morse theory. Indeed, consider a perfect HDVF that is not a DGVF, and choose a filter that induces it. Then there is no DGVF whose computation coincides with the standard homology computation.

Now a natural question arises:

Question 1 (HDVF filters)**.** *Given a HDVF, what is the set of filters induced by a given HDVF/tri-partition?*

Conjecture 3.2.8. *A filter $(\tau_i)_{i=1}^N$ induces X if and only if*

$$\tau_j \in \overline{z_q(\tau_i)} \implies j < i \quad \text{and} \quad \tau_j \in \overline{z^q(\tau_i)} \implies j > i$$

A first observation is that if F and F' are two filters such that, for each dimension q, the q-cells appear in the same relative order in F and in F', then F and F' induce the same HDVF. This follows from the equivalence between HDVFs and tri-partitions (see Propositions 2.2.4 and 2.2.5). Indeed, H. Edelsbrunner and Ölsböck 2020 shows that "the tri-partition in dimension p is invariant as long as we retain the ordering among the p-cells, rearranging the other cells at will, provided the overall ordering remains monotonic".

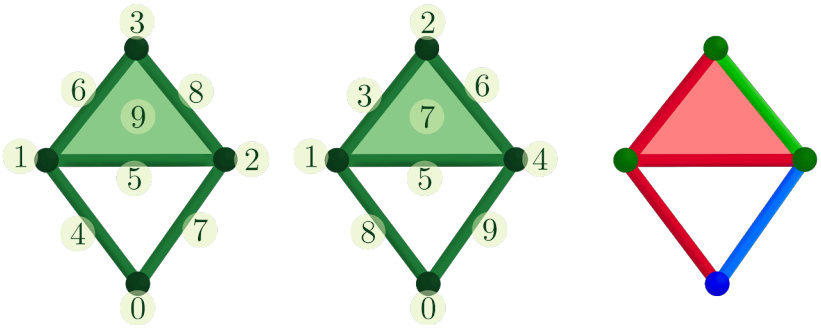


Figure 3.3 – Two different filters, represented by indices on each cell, that produce the same perfect HDVF (on the right). Note that the first filter satisfies the property of Proposition 3.2.7, and both filters satisfy the condition of Conjecture 3.2.8.

Other related prospects are related to the following hierarchy of surjections, defined for each dimension q:

filtrations $\twoheadrightarrow$ *cell orderings* $\twoheadrightarrow$[1] *perfect HDVF* $\twoheadrightarrow$ *explicit bases* $\twoheadrightarrow$ *q-homology configurations*[2]

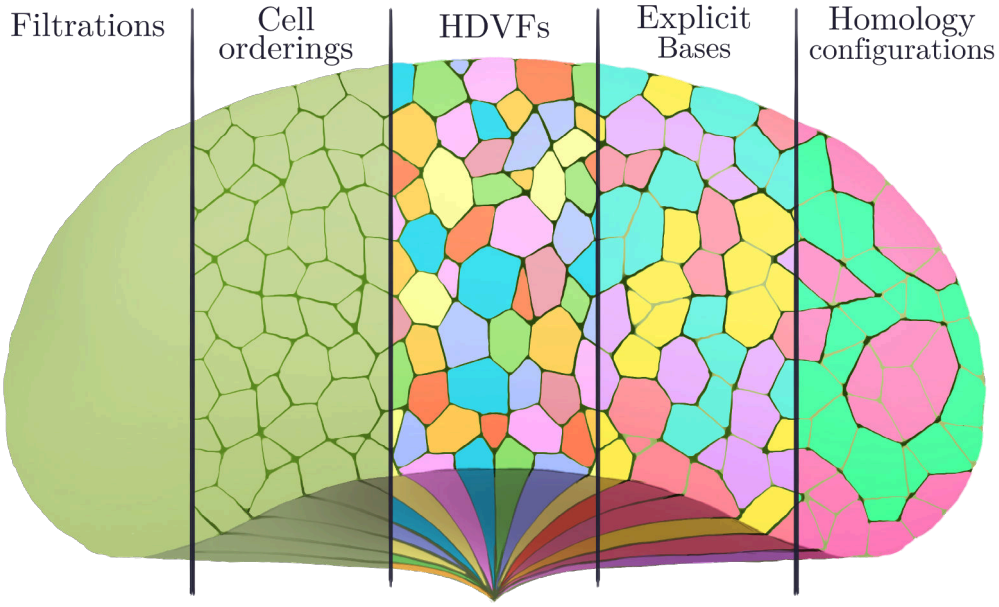


Figure 3.4 – A representation of the filtration space, partitioned differently according to different computational homology objects.

Equivalently, each object in this sequence induces a partition of the previous levels, see Figure 3.4. From a computational homology perspective, this hierarchy may help to clarify the combinatorial structure underlying persistent homology computations.

[1] To be precise, perfect HDVFs induce only a partition of strict orderings (filters), as several HDVFs can overlap on general cell orderings.
[2] introduced later in Chapter 4

We conclude with several observations that provide intuition about the filtration space. We consider filtrations of a complex with N cells.

- The set of filtrations, seen as a subset of $\mathbb{R}^N$, is convex. We call it the *filtration space*.
- A cell ordering given by a filtration (potentially with equalities) corresponds to a convex subset of the filtration space. More precisely, it defines a convex $(N-k)$-dimensional polyhedron (with possibly unbounded facets), where k is the number of equalities in the ordering. Filters correspond to N-polyhedra.
- The set of such cell orderings can naturally be equipped with a combinatorial chain complex structure. An ordering with k equalities corresponds to a cell of dimension $N-k-1$. The boundary of an ordering is defined as the formal sum of orderings obtained by introducing one additional equality. In this setting, each q-cell has exactly q faces of dimension $q-1$. This construction results in a complex whose cells correspond to orderings, see Figure 3.5. In particular, filters correspond to $(N-1)$-cells.

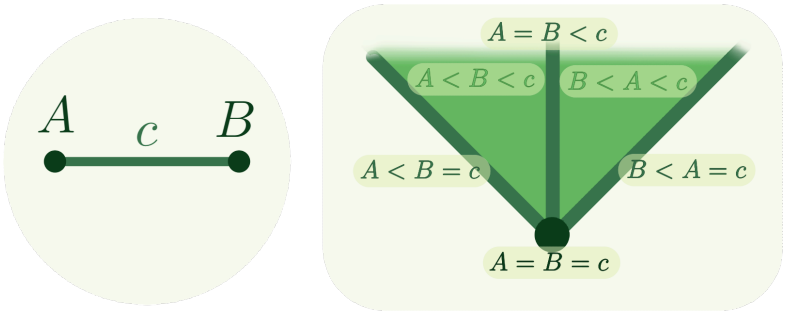


Figure 3.5 – An example of a filtration space. A simple complex with three cells (left), and the associated complex of the cell orderings of filtrations (right). "$A < B$" (resp. "$A = B$") means that the cell A arrives before B (resp. A arrives at the same time as B).

3.3. The Graph of HDVFs

HDVFs encode the computations performed by standard approaches in computational homology. Together with their elementary operations, they define a graph that represents the space of homology computations. This idea raises questions about the structure and properties of this graph. It also suggests investigating whether this graph contains information about the topology and combinatorics of the underlying complex, as well as about the associated homology computations.

3.3.1. The graph of perfect HDVFs

In this section, we study the graph of **perfect** HDVFs, whose edges are given by the three elementary operations `M`, `W`, and $\diamondsuit$.

Definition 3.3.1 (Graph of perfect HDVF)**.** Given a complex (K,∂), we define the following graph $G(K) = (V,E)$ as follows:

$$V := \mathbf{HDVF}^*(K)$$
$$E := \{(X,X') \mid \text{there exists a valid operation } \mathtt{O} \ (\mathtt{M}, \mathtt{W} \text{ or } \diamondsuit) \text{ such that } X' = \mathtt{O}X\}$$

This graph is undirected because the elementary operations are reversible. Indeed, if $X' = \mathtt{O}_\tau^\sigma X$ is valid, then $X = \mathtt{O}_\sigma^\tau X'$ is also valid.

A natural question, which has remained open since GONZALEZ-LORENZO 2016, is whether any two perfect HDVFs can be connected by a sequence of $\mathtt{M}$, $\mathtt{W}$, and $\diamondsuit$ operations. Equivalently, is the graph of Definition 3.3.1 connected[3]? This question is particularly relevant given the relationship between HDVFs and other computational homology approaches. The following theorem provides a positive answer. Recall that in this section, we only consider homology over a field $\mathscr{R}$.

Theorem 3.3.1 (connectedness of perfect HDVFs)**.** *Let X and X' be two perfect HDVFs on the same complex (K,∂). Then there exists a sequence of valid operations $\mathtt{O}_1,\dots,\mathtt{O}_m$ (each equal to $\mathtt{M}$, $\mathtt{W}$, or $\diamondsuit$) such that $X' = \mathtt{O}_m\cdots\mathtt{O}_1 X$.*

The valid operations $\mathtt{M}$, $\mathtt{W}$, and $\diamondsuit$ can be viewed as minimal operations, in the sense that they correspond to the necessary conditions for switching pairs of cells in HDVFs (see Definition 1.6.3). Theorem 3.3.1 shows that these operations are in fact sufficient to ensure the connectedness of the space of perfect HDVFs.

These questions can be put in parallel with the work of COHEN-STEINER et al. 2006, where the authors proved that the computation of persistence pairs for a given filtration can be reused for another filtration by making elementary operations. This approach provides, in particular, a proof of the stability of persistence diagrams. It has also led to several applications, including YOO et al. 2016; M. BERGOMI et al. 2020, and to the introduction of the mid-sphere, a persistent-based geometric object (see H. EDELSBRUNNER, STEPHENSON, et al. 2025).

In this section, we first give an algorithmic proof of Theorem 3.3.1. Then, we present an algorithm that adapts a perfect HDVF to a given explicit basis, and prove its correctness. Finally, we present some experimental results related to the graph of perfect HDVFs.

3.3.2. The connectedness algorithm

Let us first introduce useful notations. Given A is a set of cells, we denote $A^\perp := \overline{K}\setminus A$ the of cells not in A. Thus, Span $A^\perp$ is the space of chains that do not contain cells of A.

Given two perfect HDVFs $X = (P,S,C)$ and $X' = (P',S',C')$ on a given complex, we say that a cell is *aligned* if it has the same status in both HDVFs, i.e. if it belongs to

[3]This statement is false for the graph of general HDVFs by a simple cardinality argument: HDVFs with different numbers of primary, secondary, or critical cells in each dimension cannot be connected by these operations.

$P \cap P'$, $S \cap S'$ or $C \cap C'$. Otherwise it is considered *misaligned*. We define a distance between two such perfect HDVFs, which is designed for Algorithm 3, in particular to prove that it terminates:

$$\operatorname{dist}(X, X') = \sum_{\tau \in \overline{K}} w_{X,X'}(\tau) \qquad \text{with:}$$

$$w_{X,X'}(\tau) = \begin{cases} 0 & \text{if } \tau \text{ is aligned} \\ 1 & \text{if } \tau \text{ is misaligned and critical in } X \text{ or } X' \\ 3 & \text{otherwise, i.e. if } \tau \text{ is in } P \cap S' \text{ or } P' \cap S \end{cases}$$

Alternatively, using set notations, we can write:

$$\operatorname{dist}(X, X') := |C \cap C'^{\perp}| + |C^{\perp} \cap C'| \; + \; 3|P \cap S'| + 3|S \cap P'|$$

Intuitively, this distance counts the number of misaligned cells. Cells that are primary in one HDVF but secondary in the other are weighted by 3, to simplify the proofs. An example is shown in Figure 3.6. Note that $\operatorname{dist}(X, X') \geq 0$ for all X and X', and $\operatorname{dist}(X, X') = 0$ if and only if $X = X'$.

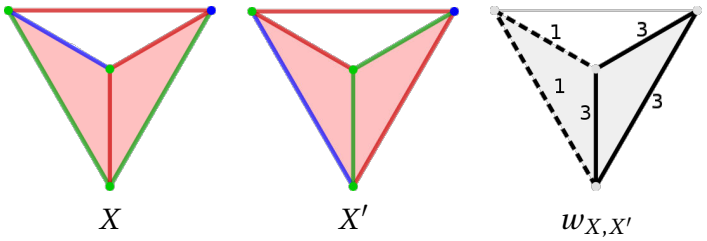


Figure 3.6 – An illustration of the distance $\operatorname{dist}(X, X')$ between two perfect HDVFs X and X'. The third image represents the weight $w_{X,X'}(\tau)$ of each cell τ. Precisely, aligned cells are shown in grey. Cells in $C \cap C'^{\perp}$ or $C^{\perp} \cap C'$ are displayed with black dashed line and contribute 1 to $\operatorname{dist}(X, X')$. Cells in $P \cap S'$ or $S \cap P'$ are displayed with solid black lines and contribute 3. The total distance is $\operatorname{dist}(X, X') = 11$.

Let us now state the three possible misalignment cases and describe the operation used for each of them to reduce this distance:

1. First, we prove that if there exists a cell $\gamma \in C \cap S'$, then there is a candidate cell $\sigma \in S$ such that $\mathtt{W}_{\sigma}^{\gamma} X$ is a valid operation that reduces the distance by at least 2. See Lemma 3.3.2.
2. Dually, we show that if there exists a cell $\gamma \in C \cap P'$, then there is a candidate cell $\pi \in P$ such that $\mathtt{M}_{\pi}^{\gamma} X$ is a valid operation that reduces the distance by at least 2. See Lemma 3.3.3.
3. Finally, if all critical cells are aligned ($C = C'$), we show that if there exists a cell $\pi \in P \cap S'$, then we can find either a candidate cell $\sigma \in S \cap P'$ such that $\diamondsuit_{\pi}^{\sigma} X$ is a valid operation that reduces the distance by 6, or a candidate cell $\gamma \in C$ such that $\mathtt{M}_{\pi}^{\gamma} X$ is a valid operation that reduces the distance by 1. See Lemma 3.3.4.

Once Lemmas 3.3.2 to 3.3.4 have been established, we can construct Algorithm 3, which connects two perfect HDVFs using $\mathtt{M}$, $\mathtt{W}$ and $\Diamond$ operations. The correctness of Algorithm 3 is guaranteed by the aforementioned lemmas, which are proved in the following subsection. An example of an execution of Algorithm 3 is given in Figure 3.7.

Algorithme 3 : Perfect HDVF connectedness algorithm

Data : $X = (P, S, C)$ and $X' = (P', S', C')$ two perfect HDVFs on (K, ∂)
Result : O: the list of operations to perform on X to obtain X'
$\delta \leftarrow \text{dist}(X, X')$;
while $\delta > 0$ **do**
  if $C \neq C'$ **then**
    $\gamma \leftarrow$ some cell in C that is misaligned;
    if $\gamma \in S'$ **then**
      $\sigma \leftarrow$ some cell in $\overline{g(\gamma)} \cap S'^{\perp}$;
      $X \leftarrow \mathtt{W}_\sigma^\gamma X$; $O.append(\mathtt{W}_\sigma^\gamma)$;
    else
      $\pi \leftarrow$ some cell in $\overline{f^*(\gamma)} \cap P'^{\perp}$;
      $X \leftarrow \mathtt{M}_\pi^\gamma X$; $O.append(\mathtt{M}_\pi^\gamma)$;
  else
    $\pi \leftarrow$ some cell in P that is misaligned;
    $\sigma \leftarrow$ some cell in $\overline{\partial h(\pi)} \cap (S \cap P')$;
    if σ *is in* $\overline{h\partial(\pi)}$ **then**
      $X \leftarrow \Diamond_\pi^\sigma X$; $O.append(\Diamond_\pi^\sigma)$;
    else
      $\gamma \leftarrow$ some cell in $\overline{f(\pi)}$;
      $X \leftarrow \mathtt{M}_\pi^\gamma X$; $O.append(\mathtt{M}_\pi^\gamma)$;
  $\delta \leftarrow \text{dist}(X, X')$;

3.3.3. Proof of the connectedness

First, let us recall the reduction diagram and the matrices of the reduction (f, g, h) induced by a HDVF (P, S, C), as we use them extensively in the following proofs.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{q+1}} & K_q & \xrightarrow{\partial_q} & K_{q-1} & \xrightarrow{\partial_{q-1}} & \cdots \\
 & \xleftarrow[h_q]{} & & \xleftarrow[h_{q-1}]{} & & \xleftarrow[h_{q-2}]{} & \\
 & & g_q \uparrow \downarrow f_q & & g_{q-1} \uparrow \downarrow f_{q-1} & & \\
\cdots & \xrightarrow{d_{q+1}} & K'_q & \xrightarrow{d_q} & K'_{q-1} & \xrightarrow{d_{q-1}} & \cdots
\end{array}$$

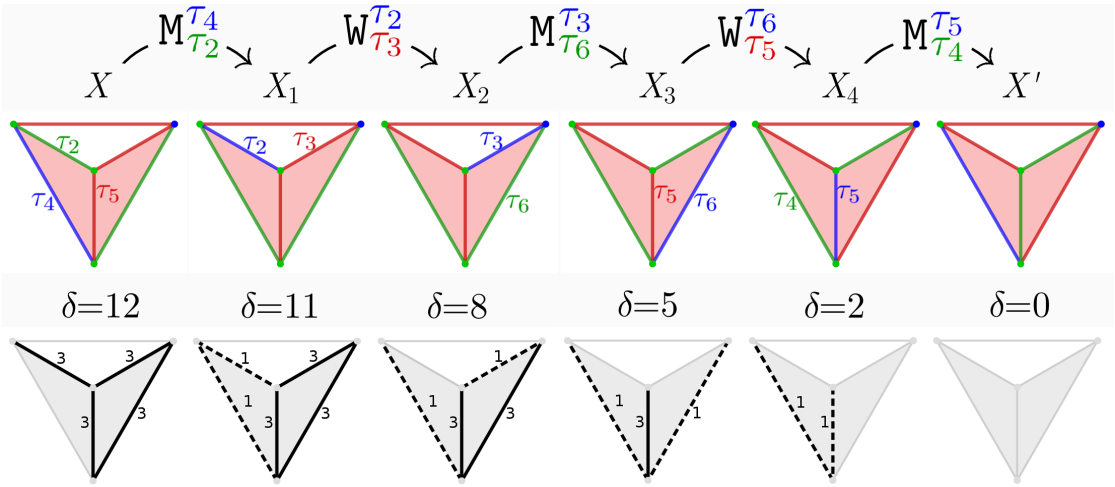


Figure 3.7 – An execution of Algorithm 3, transforming a perfect HDVF $X = (P, S, C)$ into another perfect HDVF $X' = (P', S', C')$. The top row illustrates the sequence of operations and the corresponding intermediate HDVFs, while the bottom row shows their distance $\delta = \mathrm{dist}(X_i, X')$ to X', as well as the misaligned cells at each step (represented as in Figure 3.6).
Step-by-step explanation of the first operation: initially, in X, all critical cells are aligned ($C = C'$), so we select a misaligned cell $\pi := \tau_2$ in P. Next, we choose $\sigma := \tau_5$ in $\overline{\partial h(\tau_2)} \cap (S \cap P')$. Since τ_5 is not in $\overline{h\partial(\tau_2)}$, we pick $\gamma = \tau_4$ in $\overline{f(\tau_2)}$ and perform the operation $X_1 \leftarrow \mathtt{M}_{\tau_2}^{\tau_4} X$.
For the following operations, there is always a misaligned critical cell ($C \neq C'$). Depending on whether this cell is in S' or P', we perform either a `W` or an `M` operation, respectively.

$$g := \begin{array}{c} \\ C \\ \end{array}\!\!\begin{array}{c} \\ \begin{pmatrix} 0 \\ G \\ id \end{pmatrix} \end{array}\!\!\begin{array}{l} P \\ S \\ C \end{array} \qquad f := \begin{array}{ccc} P & S & C \end{array}\!\!\begin{array}{c} \\ \begin{pmatrix} F & 0 & id \end{pmatrix} \end{array}\!\!\begin{array}{l} \\ C \end{array} \qquad h := \begin{array}{ccc} P & S & C \end{array}\!\!\begin{array}{c} \\ \begin{pmatrix} 0 & 0 & 0 \\ H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}\!\!\begin{array}{l} P \\ S \\ C \end{array} \qquad d := \begin{array}{c} C \\ \end{array}\!\!\begin{array}{c} \\ \begin{pmatrix} D \end{pmatrix} \end{array}\!\!\begin{array}{l} \\ C \end{array}$$

where $H = (\partial_S^P)^{-1}$, $F = -\partial_S^C \cdot H$, $G = -H \cdot \partial_C^P$ and $D = \partial_C^C - \partial_S^C \cdot H \cdot \partial_C^P$

We start by establishing the following three technical lemmas.

Lemma 3.3.2. *Let $X = (P, S, C)$ and $X' = (P', S', C')$ be two perfect HDVFs on the same complex. If $\gamma \in C \cap S'$, then there exists a cell σ such that $\mathtt{W}_\sigma^\gamma$ is a valid operation for X and $\mathrm{dist}(\mathtt{W}_\sigma^\gamma X, X') \leq \mathrm{dist}(X, X') - 2$. Precisely, σ can be selected in $\overline{g(\gamma)} \cap S'^{\perp}$.*

Proof. Let $\gamma \in C \cap S'$. By Lemma 1.6.3, there exists a chain $s \in \operatorname{Span} S$ such that $g(\gamma) = \gamma + s$ and $\overline{g(\gamma)} = \{\gamma\} \sqcup \bar{s}$.

Suppose $s \in \operatorname{Span} S'$. Then, as $\gamma \in S'$, we have $g(\gamma) \in \operatorname{Span} S'$. However, $\partial g(\gamma) = gd(\gamma) = 0$ because (f, g, h) is a perfect reduction. Therefore $g(\gamma) \in \ker \partial \cap \operatorname{Span} S' = \{0\}$ (by Lemma 1.6.3), which is impossible because $\gamma \neq 0$. Hence $s \notin \operatorname{Span} S'$. As a

consequence s contains a cell that is not in S'.

Let σ be a cell of $\bar{s} \cap S'^{\perp}$. We have $\sigma \in S$, moreover, we also have $\sigma \in \bar{s} \subseteq \overline{g(\gamma)}$. As a result, $\sigma \in S \cap \overline{g(\gamma)}$ so $\mathtt{W}_\sigma^\gamma$ is a valid operation for X. If we write $X^* := \mathtt{W}_\sigma^\gamma X = (P^*, S^*, C^*)$ we have the following cases:

- $\sigma \in S \cap P'$: this implies that $\sigma \in S \cap P' \cap C^*$ and $\gamma \in C \cap S' \cap S^*$. In this case:

$$w_{X^*,X'}(\sigma) = 1 = w_{X,X'}(\sigma) - 2 \qquad w_{X^*,X'}(\gamma) = 0 = w_{X,X'}(\gamma) - 1$$
$$w_{X^*,X'}(\tau) = w_{X,X'}(\tau) \quad \text{for every other cell } \tau$$

  Hence, $\mathrm{dist}(X^*, X') = \mathrm{dist}(X, X') - 3$.

- $\sigma \in S \cap C'$: this implies that $\sigma \in S \cap C' \cap C^*$ and $\gamma \in C \cap S' \cap S^*$. In this case:

$$w_{X^*,X'}(\sigma) = 0 = w_{X,X'}(\sigma) - 1 \qquad w_{X^*,X'}(\gamma) = 0 = w_{X,X'}(\gamma) - 1$$
$$w_{X^*,X'}(\tau) = w_{X,X'}(\tau) \quad \text{for every other cell } \tau$$

  Hence, $\mathrm{dist}(X^*, X') = \mathrm{dist}(X, X') - 2$.

To summarize, there exists at least one cell σ (in $\bar{s} \cap S'^{\perp} = \overline{g(\gamma)} \cap S'^{\perp}$) such that $\mathtt{W}_\sigma^\gamma$ is a valid operation for X that decreases its distance with X' by at least 2. □

Lemma 3.3.3. *Let $X = (P, S, C)$ and $X' = (P', S', C')$ be two HDVFs on the same complex. If $\gamma \in C \cap P'$, then there exists a cell π such that $\mathtt{M}_\pi^\gamma$ is a valid operation for X and $\mathrm{dist}(\mathtt{M}_\pi^\gamma X, X') \leq \mathrm{dist}(X, X') - 2$. Precisely, π can be selected in $\overline{f^*(\gamma)} \cap P'^{\perp}$.*

Proof. The proof of this lemma is the same as the last one, but considering the dual complex and the dual reduction. □

Lemma 3.3.4. *Let $X = (P, S, C)$ and $X' = (P', S', C')$ be two HDVFs on the same complex. If $C = C'$ and $\pi \in P \cap S'$, then there exists*

1. *either a cell σ such that $\Diamond_\pi^\sigma$ is valid for X and $\mathrm{dist}(\Diamond_\pi^\sigma X, X') = \mathrm{dist}(X, X') - 6$.*
2. *or a cell γ such that $\mathtt{M}_\pi^\gamma$ is valid for X and $\mathrm{dist}(\mathtt{M}_\pi^\gamma X, X') = \mathrm{dist}(X, X') - 1$.*

Precisely, in the first case, σ can be selected in $\overline{\partial h(\pi)} \cap (S \cap P')$. Otherwise, γ can be selected in $\overline{f(\pi)}$.

Proof. The proof of this lemma relies on ideas similar to those used in the two last proofs. However, it is slightly more technical as it involves two different cases.

Assume $C = C'$ and let $\pi \in P \cap S'$. We will first prove that there exist a cell τ in $S \cap P'$ that is in $\overline{\partial h(\pi)}$.

By reduction properties (see Definition 1.4.2) we have $\partial h(\pi) - \pi = -h\partial(\pi) - g f(\pi)$. The structure of HDVF reduction (see Proposition 1.6.1) implies that $\mathrm{im}(h) \subseteq \mathrm{Span}\, S$ and $\mathrm{im}(g) \subseteq \mathrm{Span}\, S \oplus \mathrm{Span}\, C = \mathrm{Span}\, P^{\perp}$. Therefore, we have $\partial h(\pi) = \pi + t$ with $t := -h\partial(\pi) - g f(\pi) \in \mathrm{Span}\, S \oplus \mathrm{Span}\, C = \mathrm{Span}\, P^{\perp}$.

As $\pi \in P$, we have $t \neq -\pi$ so $\partial h(\pi) \neq 0$. Moreover, $\partial h(\pi)$ is a boundary so Lemma 1.6.6. implies that $\overline{\partial h(\pi)} \cap P' \neq \emptyset$. In addition we have:

$$\begin{aligned}
\overline{\partial h(\pi)} \cap P' &= \overline{t} \cap P' && \text{because } \pi \notin P'.\\
&= \overline{t} \cap (S \cup C) \cap P' && \text{because } \overline{t} \subseteq S \cup C.\\
&= \left(\overline{t} \cap S \cap P'\right) \cup \left(\overline{t} \cap C \cap P'\right) &&\\
&= \overline{t} \cap S \cap P' && \text{because } C = C' \text{ so } C \cap P' = \emptyset.\\
&= \overline{\partial h(\pi)} \cap S \cap P' && \text{because } \pi \notin S \cap P'.
\end{aligned}$$

As a consequence, $\overline{\partial h(\pi)} \cap S \cap P' \neq \emptyset$. Let $\tau \in \overline{\partial h(\pi)} \cap S \cap P'$. There are two distinct cases:

- if $\tau \in \overline{h\partial(\pi)}$, then we obtain that $\tau \in S \cap \overline{\partial h(\pi)} \cap \overline{h\partial(\pi)}$, so $\diamondsuit_\pi^\tau$ is a valid operation for X. We denote $X^* := \diamondsuit_\pi^\tau X = (P^*, S^*, C^*)$.

  This operation reduces the distance with X' by 6. Indeed, $\pi \in P \cap S' \cap S^*$ and $\tau \in S \cap P' \cap P^*$ so we obtain:

$$\begin{aligned}
&w_{X^*,X'}(\pi) = 0 = w_{X,X'}(\pi) - 3 \qquad w_{X^*,X'}(\tau) = 0 = w_{X,X'}(\tau) - 3\\
&w_{X^*,X'}(\sigma) = w_{X,X'}(\sigma) \quad \text{for every other cell } \sigma
\end{aligned}$$

  Hence, $\mathrm{dist}(X^*, X') = \mathrm{dist}(X, X') - 6$.

- if $\tau \notin \overline{h\partial(\pi)}$, then we have:

$$\begin{aligned}
\langle \tau, gf(\pi)\rangle &= \langle \tau, \pi - \partial h(\pi) - h\partial(\pi)\rangle && \text{by reduction properties}\\
&= \langle \tau, \pi\rangle - \langle \tau, \partial h(\pi)\rangle - \langle \tau, h\partial(\pi)\rangle &&\\
&= -\langle \tau, \partial h(\pi)\rangle - \langle \tau, h\partial(\pi)\rangle && \text{because } \tau \text{ and } \pi \text{ are distinct cells}\\
&= -\langle \tau, \partial h(\pi)\rangle && \text{because } \tau \notin \overline{h\partial(\pi)}\\
&\neq 0 && \text{because } \tau \in \overline{\partial h(\pi)}
\end{aligned}$$

  This implies that $f(\pi) \neq 0$ so there exist a cell $\gamma \in \overline{f(\pi)}$. As $\mathrm{im}\, f \subseteq \mathrm{Span}\, C$, we have also $\gamma \in C$ so $\mathtt{M}_\pi^\gamma$ is a valid operation for X (note that we also have $\tau \in \overline{g(\gamma)}$ so $\mathtt{W}_\tau^\gamma$ is also a valid operation). We denote $X^* := \mathtt{M}_\pi^\gamma X = (P^*, S^*, C^*)$.

  This operation reduces the distance with X' by 1. Indeed, as $C = C'$, we have $\pi \in P \cap S' \cap C^*$ and $\gamma \in C \cap C' \cap P^*$ so we obtain:

$$\begin{aligned}
&w_{X^*,X'}(\pi) = 1 = w_{X,X'}(\pi) - 2 \qquad w_{X^*,X'}(\gamma) = 1 = w_{X,X'}(\gamma) + 1\\
&w_{X^*,X'}(\tau) = w_{X,X'}(\tau) \quad \text{for every other cell } \tau
\end{aligned}$$

  Hence, $\mathrm{dist}(X^*, X') = \mathrm{dist}(X, X') - 1$. □

Once the previous lemmas are shown, the proof of the correctness of Algorithm 3 is straightforward.

Proof. [Proof of Theorem 3.3.1 and correctness of Algorithm 3] Suppose $\text{dist}(X, X') > 0$, then it means at least one cell is misaligned.

- If there exists a misaligned critical cell in X (i.e., $\gamma \in C \cap C'^{\perp}$), then it must be either a primary or a secondary cell in the other HDVF. In both cases, we can apply either Lemma 3.3.3 or Lemma 3.3.2 to find a valid `M` or `W` operation that reduces the distance by at least 2.
- If all critical cells are aligned, then there must exist a cell that is primary in X but secondary in X' (otherwise, every primary cell of X would also be a primary cell of X', implying $X = X'$ because both have the same number of primary and secondary cells). In this case, we apply Lemma 3.3.4 to find a valid $\diamondsuit$ or `M` operation that decreases the distance by 6 or 1.

Hence, Algorithm 3 takes two perfect HDVFs on the same complex and modifies the first one with operations that strictly decrease the distance until it reaches zero, which means that X has gradually been changed into X'. □

3.3.4. Experimental results on the graph of HDVFs

Experimental results have been conducted in the context of the Master 1 internship of Menjatiana "Danih" ANDRIANABINITSOA, during spring 2026. Here, we only briefly present some of these experiments, please refer to ANDRIANABINITSOA et al. 2026 for a comprehensive explanation, including implementation and interpretation discussions.

The experiments were made with the CGAL HDVF package (BAC 2026), which was under development during this period.

Preliminary remarks. First, from a complexity perspective, recall that computing a perfect HDVF can be carried with a time complexity of $O(N^3)$, with N the number of cells. Performing a `M`, `W` or $\diamondsuit$ operation is in $O(N^2)$.

Second, as proved in the previous subsections, the graph of perfect HDVFs is connected. Moreover, the algorithm to connect two perfect HDVFs also works per dimension. Following from the results of Section 2.2.2, this implies that q-tri-partitions are also connected under `M`, `W` and $\diamondsuit$ operations.

In addition, as mentioned in Section 2.2.2, perfect HDVFs are in one to one correspondence with sequences $(X_0, \dots, X_n)$ of q-tri-partitions X_q. This implies that the graph of perfect HDVFs is the Cartesian product of the graphs of q-tri-partitions, $q = 1, \dots, n$. As a consequence, it is relevant to study the graph of perfect HDVFs at a given dimension q (i.e. the graph of q-tri-partitions), instead of considering the whole graph of perfect HDVFs. Indeed, the whole graph is much bigger and does not carry more information.

The number of HDVFs and tri-partitions. In order to carry out experiments on the graph of HDVFs, we use series of small simplicial complexes, showing a range of

Betti numbers and simplices organizations. Figure 3.8 introduces these complexes.

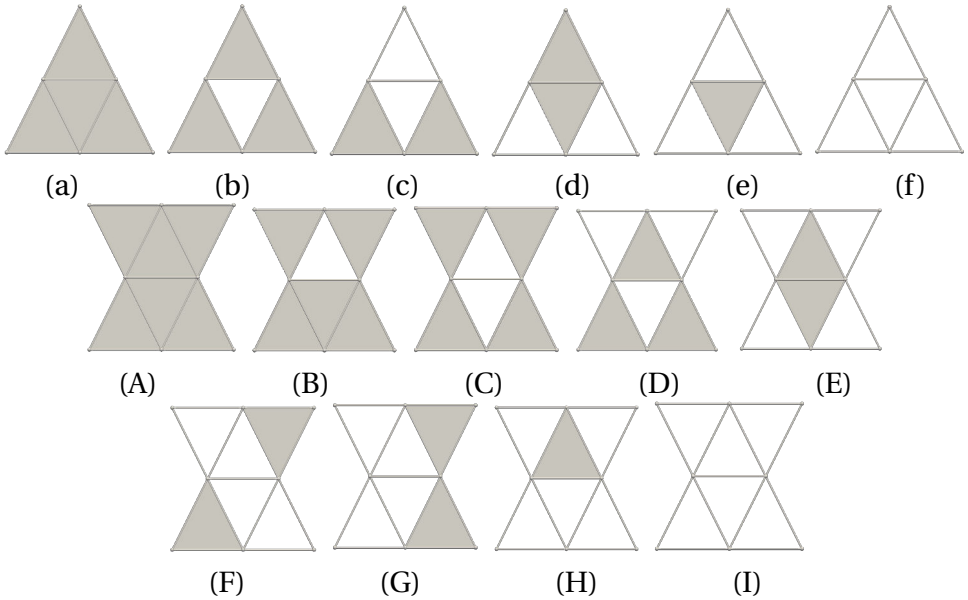


Figure 3.8 – Complexes used for experiments: (top) a series of small complexes based on four triangles, (middle, bottom) complexes based on six triangles.

We compute the graph of perfect HDVFs for dimension 1 (i.e. the graph of 1-tri-partitions) for each complex and evaluate simple measures on the graph: number of nodes ($|HDVF_1|$), diameter and *minimum outgoing path* (denoted by MOP: the minimum, over all nodes, of their distance to their farthest neighbor).

We compare the size of this graph to a combinatorial upper bound on the number of 1-tri-partitions denoted $\mathscr{B}_1$. This bound is the number of partitions of the 1-cells into three categories, whose cardinals agree with the cardinals of primary, secondary and critical cells given by HDVFs. Precisely, we have $\mathscr{B}_1 = \binom{N_1}{\beta_1} \cdot \binom{N_1-\beta_1}{N_0-\beta_0}$, with N_q the number of q-cells.

Table 3.2 relates measures of complexes (total number of simplices $|\overline{K}|$, and Betti numbers) and measures of this graph: number of nodes ($|HDVF_1|$), diameter and minimum outgoing path (MOP).

The large difference between $|HDVF_1|$ and $\mathscr{B}_1$ highlights the strength of the constraint in the definition of HDVF (the invertibility of ∂_S^P). Indeed, most of the partitions of 1-cells in three categories, with appropriate cardinals, do not come from a HDVF.

The diameter is very small compared to the total number of HDVFs. Moreover, the difference between the diameter and the MOP is small. This suggests that no HDVF has a "central" position in the graph, and that no homology computation is "more generic". For a more detailed discussion, please refer to ANDRIANABINITSOA et al. 2026, which presents additional experiments and measures, for instance the repartition of degrees.

	$\lvert\overline{K}\rvert$	β_1	$\lvert HDVF_1\rvert$	$\mathscr{B}_1$	diameter	MOP
a	19	0	54	126	4	3
b	18	1	108	504	4	4
c	17	2	90	756	4	4
d	17	2	90	756	4	4
e	16	3	90	504	4	4
f	15	4	54	126	4	3

	$\lvert\overline{K}\rvert$	β_1	$\lvert HDVF_1\rvert$	$\mathscr{B}_1$	diameter	MOP
A	27	0	360	1716	6	4
B	26	1	828	10296	6	5
C	25	2	1008	25740	6	6
D	24	3	996	34320	6	6
E	23	4	880	25740	6	6
F	23	4	648	25740	6	6
G	23	4	648	25740	6	6
H	22	5	624	10296	6	5
I	21	6	360	1716	6	4

Table 3.2 – Statistics on the space of HDVFs for complexes of figure 3.8.

Connectivity and shortest paths. We present a second experiment: a comparison between the *connectedness length* (length of paths computed by Algorithm 3) with the length of the shortest path in the graph of HDVFs. The difference between these two measures is the *error*, denoted by δ. Precisely, given a perfect HDVF X, δ is the maximal error over pairs (X, X'), with X' another perfect HDVF.

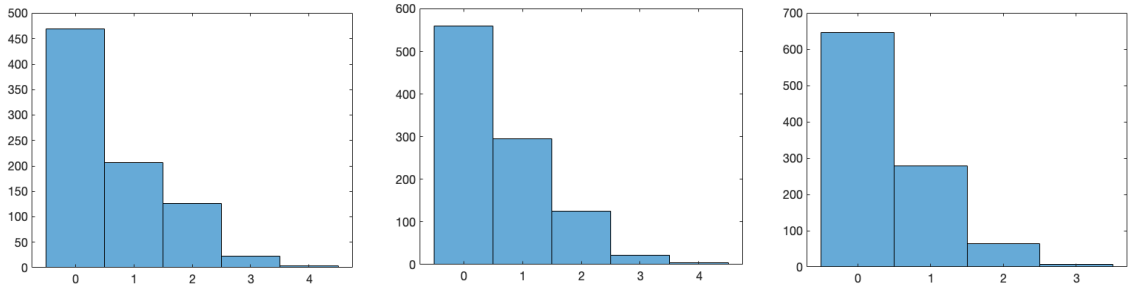
> *Remark.* The connectedness length is different from the *alignment distance* $\mathrm{dist}(\cdot,\cdot)$ defined in Section 3.3. For instance, in Figure 3.7, the two extremal HDVFs have connectedness length 5 but alignment distance 12.

Table 3.3 give the maximum value of δ for various complexes and the number ($n_{\max}$) of HDVFs reaching the maximal error. Figure 3.9 shows some histograms for the values of δ. It is important to note that these data depends on the algorithm implementation, as different choices of cell can be made during Algorithm 3. Hence, a different implementation (or even a different cell numbering) may result in different data.

Besides their potential variability, these figures shows that the path computed by the Algorithm 3 is usually close to the shortest path. Indeed, in our examples, the large majority of HDVFs produce a δ below 2, in a graph with a diameter equals to 6. However, in some rare cases, δ can raise up to 4. This is partly due to arbitrary choices of cell, but also because Algorithm 3 greedily favors `M` and `W` operations, while shorter paths maybe be obtained by $\Diamond$ operations, as illustrated in Figure 3.10.

	$\lvert\overline{K}\rvert$	β_1	$\lvert HDVF_1\rvert$	diameter	max(δ)	$n_{\max}$
A	27	0	360	6	0	
B	26	1	828	6	4	4
C	25	2	1008	6	4	4
D	24	3	996	6	3	8
E	23	4	880	6	2	32
F	23	4	648	6	2	26
G	23	4	648	6	2	8
H	22	5	624	6	1	72
I	21	6	360	6	0	

Table 3.3 – Error δ for previous complexes.



(B) (C) (D)

Figure 3.9 – Distribution of the error δ for complexes where this error is greater than 2. The horizontal axis represents δ and the vertical axis is the number of HDVFs achieving a specific δ value.

3.4. Adapting a Perfect HDVF to a Given Explicit Basis

Chapter 2 establishes a correspondence between HDVFs and explicit homology bases, see Theorem 2.3.5. With the graph of perfect HDVFs in mind, this leads us to the following question: is it possible to adapt a given HDVF to a given explicit basis? Theorem 3.3.1 implies an affirmative answer: it is sufficient to build a perfect HDVF associated with the given explicit basis, and then apply the connectedness algorithm to transform the given HDVF into this new HDVF. However, this is an inefficient algorithm with a cubic complexity.

This question is particularly relevant in settings where a HDVF has already been computed, and its associated homology basis is later modified, for instance using geometric heuristics. In such cases, one would prefer to adapt the existing HDVF to the new basis, rather than recomputing a HDVF from scratch.

In this section, we therefore address the problem of adapting a perfect HDVF to a given explicit basis using the operations `M`, `W`, and $\diamondsuit$, with a better complexity. For simplicity, this section considers homology over the field $\mathbb{Z}/2\mathbb{Z}$. Yet, all the results hold

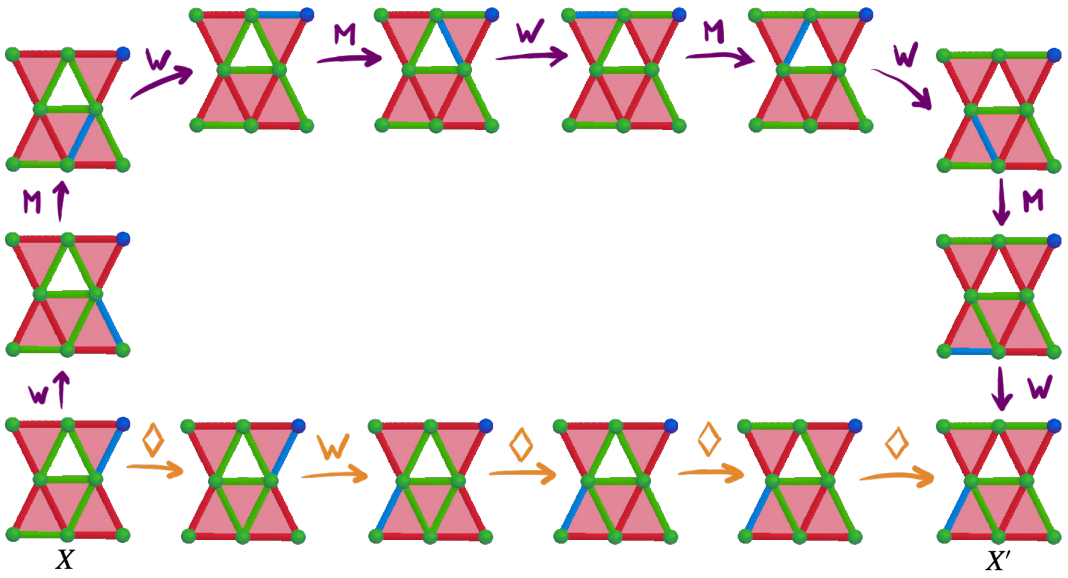


Figure 3.10 – Example of large divergence between the shortest path from an HDVF X to an HDVF X' (orange arrows), of length 5, and the path produced by Algorithm 3 (purple arrows), of length 9.

for other fields, with eventually non-zero coefficients in front of generators.

3.4.1. The adaptation algorithm

Let $X = (P, S, C)$ be a perfect HDVF on (K, ∂) and $\{\mathrm{g}_i\}_{i \leq \beta_q}$ be an explicit q-homology basis. Our goal is to transform X using operations `M`, `W`, and $\Diamond$ into a new HDVF that induces this basis. More precisely, we search for a sequence of operations $O_1, \ldots, O_k$ such that the perfect HDVF $(P', S', C') = O_k \cdots O_1 X$, with associated reduction (f', g', h'), satisfies $\{g'(\gamma)\}_{\gamma \in C'_q} = \{\mathrm{g}_1, \ldots, \mathrm{g}_{\beta_q}\}$.

We propose an algorithm with time complexity $O(m \cdot N^2)$ to achieve this task, where m is the number of q-cells in the basis ($m = |\cup_{i=1}^{\beta_q} \overline{\mathrm{g}_i}| = |\overline{K_q^{\downarrow \mathrm{g}}}|$) and N is the total number of q-cells ($N = |\overline{K_q}|$).

Recall that $K^{\downarrow \mathrm{g}}$ is the combinatorial chain subcomplex of K induced by the cells that appear in the explicit basis together with all their facets. This algorithm is divided in three steps (see Algorithm 4 for details, and see Figure 3.11 for an example):

1. First, apply `M` operations to move every critical q-cell of X into $K_q^{\downarrow \mathrm{g}}$.
2. Next, apply $\Diamond$ and `M` operations to move the remaining primary cells outside of $K_q^{\downarrow \mathrm{g}}$.
3. Finally, apply `W` operations to move the critical q-cells to their respective locations, that is, move each γ into $\overline{\mathrm{g}_i} \setminus \bigcup_{j \neq i} \overline{\mathrm{g}_j}$ for some i.

The correctness of the *Step 1* is a consequence of Proposition 3.4.2. After this step, every critical cell is in $K^{\downarrow \mathrm{g}}$. Hence, Proposition 3.4.3 proves the correctness of *Step 2*. Finally, after *Step 2*, $K_q^{\downarrow \mathrm{g}}$ is composed of secondary and all critical q-cells,

Algorithme 4 : Adapting a HDVF to an explicit basis

Data : A perfect HDVF $X = (P, S, C)$ for a complex (K, ∂) and an explicit q-homology basis $\{\mathrm{g}_i\}_{i \leq \beta_q}$.

Result : A perfect HDVF X whose induced q-homology basis is $\{\mathrm{g}_i\}_{i \leq \beta_q}$.

`/* Step 1. Bring the critical cells in` $K_q^{\downarrow \mathrm{g}}$ `*/`

for *critical cell* γ *not in* $K_q^{\downarrow \mathrm{g}}$ **do**

- $\pi \leftarrow$ cell in $\overline{f^*(\gamma)} \cap \overline{K}_q^{\downarrow \mathrm{g}}$;
- $X \leftarrow \mathtt{M}_\pi^\gamma X$;

`/* Step 2. Get primary cells out of` $K_q^{\downarrow \mathrm{g}}$ `*/`

while $P \cap \overline{K}_q^{\downarrow \mathrm{g}} \neq \emptyset$ **do**

- $\pi \leftarrow$ cell in $P \cap \overline{K}_q^{\downarrow \mathrm{g}}$;
- $\sigma \leftarrow$ cell in $\overline{\partial h(\pi)} \setminus \overline{K}_q^{\downarrow \mathrm{g}}$;
- **if** $\langle z_q(\pi), \sigma \rangle \neq 0$ **then**
  - $X \leftarrow \diamondsuit_\pi^\sigma X$;
- **else**
  - $\gamma \leftarrow$ cell in $\overline{f(\pi)} \cap \overline{g^*(\sigma)}$;
  - $X \leftarrow \diamondsuit_\gamma^\sigma \mathtt{M}_\pi^\gamma X$;

`/* Step 3. Get critical cells to their generators */`

for *critical cell* γ *such that* $\gamma \in \overline{\mathrm{g}_i} \cap \overline{\mathrm{g}_j}$ *for some* i *and* j **do**

- $\sigma \leftarrow$ cell in $\overline{g(\gamma)}$ and in $\overline{\mathrm{g}_k} \setminus \bigcup_{j \neq k} \overline{\mathrm{g}_j}$ for some k;
- $X \leftarrow \mathtt{W}_\sigma^\gamma X$;

therefore Proposition 3.4.4 proves the correctness of *Step 3*. At the end of the algorithm, we obtain a perfect HDVF (P', S', C') where every g_i is composed of a unique critical cell γ_i and secondary cells. Lemma 1.6.3 ensures that we indeed have $g'(\gamma_i) = \mathrm{g}_i$, so $\{g'(\gamma)\}_{\gamma \in C'_q} = \{\mathrm{g}_1, \dots, \mathrm{g}_{\beta_q}\}$.

To prove the correctness of the different steps, we need the following lemma about explicit bases:

Lemma 3.4.1. *If* $\{\mathrm{g}_i\}_{i \leq \beta_q}$ *is an explicit* q*-homology basis, then* $\operatorname{im}\partial \cap K_q^{\downarrow \mathrm{g}} = \{0\}$.

Proof. First, a direct consequence of Proposition 2.3.1(2) is that if $\{\mathrm{g}_i\}_{i \leq \beta_q}$ is an explicit q-homology basis we have:

$$\ker\partial \cap K_q^{\downarrow \mathrm{g}} \subseteq \operatorname{Span}\left(\{\mathrm{g}_i\}_{i \leq \beta_q}\right) \tag{3.1}$$

Let $x \in \operatorname{im}\partial \cap K_q^{\downarrow \mathrm{g}}$. As $\operatorname{im}\partial \subseteq \ker\partial$, Equation (3.1) implies that $x \in \operatorname{Span}\left(\{\mathrm{g}_i\}_{i \leq \beta_q}\right)$, so there exist coefficients $(\varepsilon_i)_{i \leq \beta_q}$ such that $x = \sum_{i=1}^{\beta_q} \varepsilon_i \mathrm{g}_i$. By looking at the homology classes in K, this implies that $[x]_q^K = \sum_{i=1}^{\beta_q} \varepsilon_i [\mathrm{g}_i]_q^K$. However, $[x] = 0$ (because $x \in \operatorname{im}\partial$) and $\{\mathrm{g}_i\}_{i \leq \beta_q}$ is a homology basis so this implies that $\forall i, \varepsilon_i = 0$. Thus, $x = 0$. □

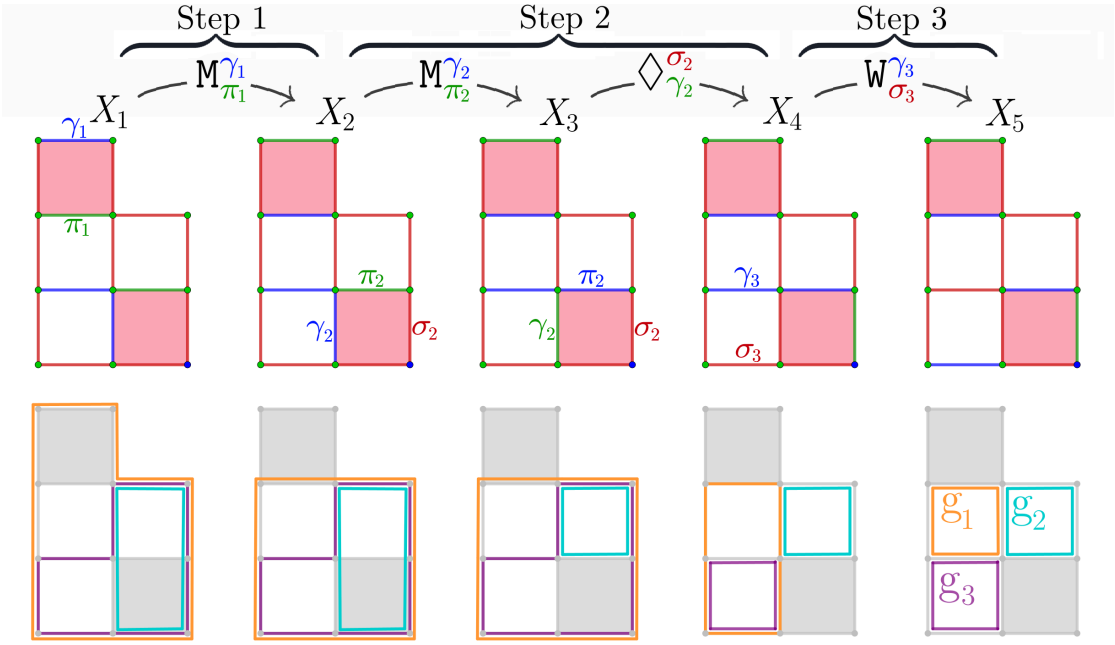


Figure 3.11 – An example of the adaptation of a perfect HDVF X_1 to the 1-homology basis $\{g_1, g_2, g_3\}$ using Algorithm 4. The first row represents the consecutive HDVFs, while the second row represents their corresponding 1-homology bases. In *Step 1*, we want all critical 1-cells to be in $K_1^{\downarrow g}$: we perform $X_2 \leftarrow \mathtt{M}_{\pi_1}^{\gamma_1} X_1$. In *Step 2*, we want to get primary cells out of $K_1^{\downarrow g}$: we choose $\pi_2 \in P \cap \overline{K}_1^{\downarrow g}$ and $\sigma_2 \in \overline{\partial h(\pi)} \setminus \overline{K}_1^{\downarrow g}$. As $\langle z_1(\pi_2), \sigma_2 \rangle = 0$, we are in the second case of *Step 2*: we choose $\gamma_2 \in \overline{f(\pi_2)} \cap \overline{g^*(\sigma_2)}$ and perform the ternary operation $X_4 \leftarrow \Diamond_{\gamma_2}^{\sigma_2} \mathtt{M}_{\pi_2}^{\gamma_2} X_2$. In *Step 3*, $K_1^{\downarrow g}$ is composed of only secondary and critical cells, but we want one critical cell per g_i: we perform the operation $X_5 \leftarrow \mathtt{W}_{\sigma_3}^{\gamma_3} X_4$.

3.4.2. Proof of the correctness of the adaptation

As for the proof of the connectedness algorithm, reduction properties and HDVF reduction matrices properties are used extensively in the following proofs.

Step 1. Here, we prove the correctness of *Step 1*. Indeed, the following proposition implies that it is possible to perform $\mathtt{M}$ operations to put every critical cell in $K^{\downarrow g}$.

Proposition 3.4.2. *If γ is a critical q-cell that is not in $K_q^{\downarrow g}$, then there exists a primary cell π in $K_q^{\downarrow g}$ such that $\mathtt{M}_\pi^\gamma$ is a valid operation.*

Proof. Consider a perfect HDVF $X = (P, S, C)$ with reduction (f, g, h) such that γ is a cell of C_q that is not in K_q^β. By Theorem 2.3.5, let $X' = (P', S', C')$ be a perfect HDVF with reduction (f', g', h') such that $\{g'(\gamma)\}_{\gamma \in C'_q} = \{g_i\}_{i \leq \beta_q}$.

First we have $fg'f'g = id$. Indeed, using perfect reduction properties we get:

$$\begin{aligned} fg'f'g = f(id - \partial h' - h'\partial)g &= fg - f\partial h'g - fh'\partial g \\ &= id - 0 - 0 \qquad \text{because } fg = id,\ f\partial = 0 \text{ and } \partial g = 0 \end{aligned}$$

Denote $x := g'f'g(\gamma)$, this implies that $f(x) = \gamma$. Hence, let $\pi \in \overline{x}$ such that $\langle f(\pi), \gamma\rangle \neq 0$. As $f \circ \mathtt{i}_S = 0$ and $f \circ \mathtt{i}_C = id$, π is either a primary cell or is equal to γ.

We have $x \in \operatorname{im} g'_q$ so $x \in \operatorname{Span}(\{\mathtt{g}_i\}_{i \leq \beta_q}) \subseteq K_q^\beta$. This implies that π is in K_q^β, so it cannot be equal to γ. As a result, π is a primary cell in K_q^β such that $\langle f(\pi), \gamma\rangle \neq 0$ i.e. $\mathtt{M}_\pi^\gamma$ is a valid operation for X □

Step 2. To prove the correctness of *Step 2*, we need to use ternary operations (see Lemma 1.6.9). The following proposition proves the correctness of *Step 2*, under the assumption that every critical cell is in $K^{\downarrow \mathtt{g}}$, which is ensured by *Step 1* (see Proposition 3.4.2).

Proposition 3.4.3. *Let $X = (P, S, C)$ be a perfect HDVF for (K, ∂) and $\{\mathtt{g}_i\}_{i \leq \beta_q}$ be an explicit q-homology basis with associated sub-complex $(K^{\downarrow \mathtt{g}}, \partial)$. Suppose $C_q \subseteq \overline{K}^{\downarrow \mathtt{g}}$ and $\pi \in P_q \cap \overline{K}^{\downarrow \mathtt{g}}$. Then, at least one of the two statements is true:*

- *there exists $\sigma \in S_q \backslash \overline{K}^{\downarrow \mathtt{g}}$ such that $\Diamond_\pi^\sigma$ is valid for X.*
- *there exists $\sigma \in S_q \backslash \overline{K}^{\downarrow \mathtt{g}}$ and $\gamma \in C_q$ such that $\Diamond_\gamma^\sigma \mathtt{M}_\pi^\gamma$ is valid for X.*

Precisely, it is possible to choose σ in $\overline{\partial h(\pi)}$ and eventually γ in $\overline{f(\pi)} \cap \overline{g^*(\sigma)}$. In both cases, the operation moves a primary cell out of $K^{\downarrow \mathtt{g}}$.

Proof. By reduction properties we have:

$$\begin{aligned} gf(\pi) &= \pi - h\partial(\pi) - \partial h(\pi) \\ \partial h(\pi) &= z_q(\pi) - gf(\pi) \end{aligned}$$

By contradiction, suppose that $\partial h(\pi) \in K^\beta$, then it belongs to $\operatorname{im}\partial \cap K^\beta$ so it equals to zero by Lemma 3.4.1. Hence, we get that $z_q(\pi) = gf(\pi)$, which is not possible because $\langle z_q(\pi), \pi\rangle = 1$ but $\langle gf(\pi), \pi\rangle = 0$ (as $\pi \in P$ and $\operatorname{im} g \in \operatorname{Span} P^\perp$).

Therefore, $\overline{\partial h(\pi)} \backslash \overline{K^\beta} \neq \emptyset$.

Let $\sigma \in \overline{\partial h(\pi)} \backslash \overline{K^\beta}$. We have by definition $\overline{z_q(\pi)} \subseteq \{\pi\} \sqcup S_q$ and $\overline{gf(\pi)} \subseteq C_q \sqcup S_q$. As $\partial h(\pi) = z_q(\pi) - gf(\pi)$, we have $\overline{\partial h(\pi)} \subseteq \{\pi\} \sqcup C_q \sqcup S_q$. However, π and C_q are inside $\overline{K^\beta}$ by assumption, so $\overline{\partial h(\pi)} \backslash \overline{K^\beta} \subseteq S_q$, which implies that $\sigma \in S_q$.

As $\partial h(\pi) = z_q(\pi) - gf(\pi)$, we distinguish two cases:

- σ is in $\overline{z_q(\pi)}$: first, $\sigma \in \overline{\partial h(\pi)}$ so $\langle \partial h(\pi), \sigma\rangle \neq 0$. Second, as σ is secondary and $z_q = id - h\partial$, we obtain that $\langle h\partial(\pi), \sigma\rangle \neq 0$. Therefore, by Definition 1.6.3, $\Diamond_\pi^\sigma$ is a valid operation for X and $\sigma \in S_q \backslash \overline{K^\beta}$.

- σ is in $\overline{g\,f(\pi)}$: first, $\sigma \in \overline{\partial h(\pi)}$ so $\langle \partial h(\pi), \sigma \rangle \neq 0$. second, $\sigma \in \overline{g\,f(\pi)}$ implies that there exists $\gamma \in \overline{f(\pi)} \subseteq C_q$ such that $\langle g(\gamma), \sigma \rangle \neq 0$. As $\gamma \in \overline{f(\pi)}$, we also have $\langle f(\pi), \gamma \rangle \neq 0$. Therefore, by Lemma 1.6.9, $\Diamond_\gamma^\sigma \mathtt{M}_\pi^\gamma$ is a valid operation for X, with $\sigma \in S_q \backslash \overline{K^\beta}$ and $\gamma \in C_q$.

□

Step 3. Finally, the following proposition ensures that *Step 3* is correct under the assumption that *Step 2* was performed.

Proposition 3.4.4. *Suppose that $C_q \subseteq K_q^{\downarrow \mathrm{g}} \subseteq \operatorname{Span} P^\perp$ (i.e. every critical q-cell is in the basis complex and its q-cells are either critical or secondary). Let $\gamma \in C_q$ be a critical cell that is in $\overline{\mathrm{g}_i} \cap \overline{\mathrm{g}_j}$ for some $i \neq j$.*

Then there exists k and a secondary cell σ in $\overline{\mathrm{g}_k} \backslash \bigcup_{j \neq k} \overline{\mathrm{g}_j}$ such that $\mathtt{W}_\sigma^\gamma$ is a valid operation. This ensures that we can perform Step 3 of Algorithm 4.

Proof. First, let us show that $g(\gamma)$ is in $\operatorname{Span}(\{\mathrm{g}_i\}_{i \leq \beta_q})$. As $\{\mathrm{g}_i\}_{i \leq \beta_q}$ is a q-homology basis and $g(\gamma)$ is a q-cycle, there exists coefficients ε_j and a $q+1$-chain y such that: $g(\gamma) - \sum_{j=1}^{\beta_q} \varepsilon_j \mathrm{g}_j = \partial(y)$.

Every g_j is in K_q^β, so $\sum_{j=1}^{\beta_q} \varepsilon_j \mathrm{g}_j \in \operatorname{Span} P^\perp$. As $g(\gamma) \in \gamma + \operatorname{Span} S \subseteq \operatorname{Span} P^\perp$ we obtain that $\partial(y) \in \operatorname{im}\partial \cap \operatorname{Span} P^\perp$. This implies that $\partial(y) = 0$, otherwise we would have a contradiction (see Lemma 1.6.6). As a result:

$$g(\gamma) = \sum_{j=1}^{\beta_q} \varepsilon_j \mathrm{g}_j$$

Now, As $g(\gamma) \neq 0$, there exists k such that $\varepsilon_k \neq 0$. Let $\sigma \in \overline{\mathrm{g}_k} \backslash \bigcup_{j \neq k} \overline{\mathrm{g}_j}$ (which is not empty by Definition 2.3.2). We have:

$$\begin{aligned} \langle g(\gamma), \sigma \rangle &= \varepsilon_k \langle \mathrm{g}_k, \sigma \rangle + \sum_{j \neq k} \varepsilon_j \langle \mathrm{g}_j, \sigma \rangle \\ &= \varepsilon_k \langle \mathrm{g}_k, \sigma \rangle && \text{as } \langle \mathrm{g}_j, \sigma \rangle = 0 \text{ if } j \neq k \\ &\neq 0 && \text{as both } \varepsilon_k \text{ and } \langle \mathrm{g}_k, \sigma \rangle \text{ are non-zero} \end{aligned}$$

As $\sigma \in \overline{g(\gamma)}$ and $g(\gamma) \in \gamma + \operatorname{Span} S$, σ is either a secondary cell or is equal to γ. However, as $\gamma \in \overline{\mathrm{g}_i} \cap \overline{\mathrm{g}_j}$ (for some $i \neq j$) and $\sigma \in \overline{\mathrm{g}_k} \backslash \bigcup_{j \neq k} \overline{\mathrm{g}_j}$, σ cannot be equal to γ.

To sum it up, $\sigma \in \overline{\mathrm{g}_k} \backslash \bigcup_{j \neq k} \overline{\mathrm{g}_j}$ is a secondary cell such that $\langle g(\gamma), \sigma \rangle \neq 0$: $\mathtt{W}_\sigma^\gamma$ is a valid operation. □

Time complexity. The complexity of the algorithm is $O(m \cdot N^2)$, with $m = |\overline{K}_q^{\downarrow \mathrm{g}}|$ and $N = |\overline{K_q}|$. Indeed, *Step 1* costs at most β operations $\mathtt{M}$, as there are no more than β critical cells. *Step 2* costs at most $2m$ operations ($\mathtt{M}$ or $\Diamond$), as it removes a cell

from $P \cap \overline{K}_q^{\lfloor g}$ at each iteration using at most two operations. *Step 3* costs at most β operations `W`. Every operation `M`, `W` or $\diamondsuit$ costs $O(N^2)$ time, therefore, we obtain a total complexity of $O(m \cdot N^2 + 2\beta \cdot N^2)$. Note that we always have $m \geq \beta$ because of the first condition of Definition 2.3.2.

3.5. Conclusion

Persistent homology provides a relevant framework for studying the evolution of homological features across a filtration. It was shown in GONZALEZ-LORENZO, BAC, MARI, et al. 2017 that a filtration induces a unique perfect HDVF, which contains its persistent homology information. In this chapter, we firstly refine this result by proving that each step of Algorithm 2 coincides with the corresponding step of the standard persistence algorithm Algorithm 1 from EDELSBRUNNER et al. 2002.

We also investigate the converse problem of building a filtration that induces a given perfect HDVF. Although this question is not fully resolved (see Conjecture 3.2.8), we show that a given HDVF can be induced by multiple filtrations and provide an explicit construction of such filtrations. As a consequence, given an explicit homology basis, we can build a filtration that induces this basis, either in cubic time from scratch or in quadratic time when a perfect HDVF is already available.

These results highlight that perfect HDVFs, explicit bases, and *homology configurations* (introduced in Chapter 4) form successive partitions of the filtration space. This suggests that HDVFs provide a framework for analyzing the space of filtrations and their associated persistent homology. These ideas are related to COHEN-STEINER et al. 2006, which studies how persistence pairings evolve when two consecutive cells are swapped. This point of view motivates several open questions.

- Can we define operations on filtrations that correspond to the operations `M`, `W`, and $\diamondsuit$ on HDVFs?
- Could such operations be used to compute and analyze vines and vineyards COHEN-STEINER et al. 2006, for instance to study time-varying data as in M. BERGOMI et al. 2020; YOO et al. 2016?
- Similarly to Question 1, can we characterize the set of filtrations that induce a given explicit basis, or more generally a given homology configuration?
- Given a filtration F and an explicit basis $\{\mathrm{g}_i\}_{i \leq \beta_q}$, can one compute the closest filtration to F that induces this basis? Algorithm 4 might be a good start.

In a second time, we study the space of HDVFs by equipping the set of perfect HDVFs with a graph structure based on the operations `M`, `W`, and $\diamondsuit$. We derive an algorithm that finds a sequence of operations to link two given perfect HDVFs, hence proving the connectedness of this graph. We also build an algorithm that navigates this graph to adapt a given perfect HDVF to a given explicit basis, with quadratic time complexity.

Since the operations `M`, `W`, and $\diamondsuit$ are defined independently in each dimension (see Definition 1.6.3), they naturally extend to tri-partitions. As a consequence, the

connectivity and adaptation algorithms for perfect HDVFs directly translate to tri-partitions.

Proposition 3.5.1. *The space of q-tri-partitions over a complex K is connected under elementary operations* `M`, `W` *and* $\diamondsuit$.

Proposition 3.5.2. *Algorithm 4 also adapts a q-tri-partition to a given explicit q-homology basis.*

Then, we use a preliminary version of the HDVF package of the CGAL library (see BAC 2026), to realize experiments on the graph of HDVFs. For instance, we compute the size and the diameter of this graph for different complexes. The results suggest that in general, the graph is "hollow", without any central HDVF nor hub. The HDVF package will soon be available in the CGAL library, we hope it will provide further experiments and help to address several open questions.

- Generally, what is the shape of the graph of perfect HDVFs for a fixed complex? How does this graph compare to the graph whose vertices are partitions of cells into three categories (not necessarily producing a HDVF), and whose edges are exchange operations in between categories?
- Given a non-explicit homology basis $\{g_i\}_{i\leq\beta_q}$, can we define and compute a "closest" explicit basis?
- How can we adapt a perfect HDVF to an incomplete explicit basis?

Finally, this chapter opens the broader question of how homology computations relate to each others. A relevant prospect would be to study HDVFs and tri-partitions from a *matroid* point of view. While we did not investigate this idea, some preliminary results about the matroid structure of the set of tri-partitions can be found in H. EDELSBRUNNER and ÖLSBÖCK 2020.

4. Beyond Standard Homology Information

Sommaire

4.1. Introduction . . . 115
4.2. A Constructive Approach to Alexander Duality . . . 117
4.2.1. Relative and reduced homology . . . 117
4.2.2. Combining complexes and HDVFs . . . 119
4.2.3. The combinatorial duality . . . 123
4.2.4. Combinatorial Alexander duality for geometric complexes . . . 127
4.2.5. Singular Alexander duality for simplicial complexes . . . 130
4.2.6. Homology quartets . . . 139
4.3. Homology Configurations . . . 142
4.3.1. Motivations . . . 142
4.3.2. Definition and intuition . . . 143
4.3.3. Properties . . . 145
4.3.4. Combinatorial invariance . . . 150
4.3.4.1. General and elementary subdivisions . . . 151
4.3.4.2. Plan of the proof of combinatorial invariance . . . 153
4.3.4.3. Building an injection from $\mathbf{Conf}_n(K)$ to $\mathbf{Conf}_n(K')$ . . . 154
4.3.4.4. Building an injection from $\mathbf{Conf}_n(K')$ to $\mathbf{Conf}_n(K)$ . . . 157
4.3.5. Topological invariance and the Hauptvermutung . . . 162
4.3.6. Computing homology configurations . . . 164
4.4. About the Constructiveness of Homology Morphisms . . . 169
4.4.1. Constructible homology-cohomology isomorphisms . . . 169
4.4.2. Constructible combinatorial duality isomorphisms . . . 170
4.5. Conclusion . . . 177

4.1. Introduction

Topology studies objects up to continuous deformations; hence, a natural question is the link between the topology of an object and that of its complement. This question, usually called duality, has long been at the heart of many works. Its earliest form was established for oriented closed manifolds by H. Poincaré in 1893, who stated (without proof) that, given such an n-manifold, for any $q \leq n$, the qth and $(n-q)$th Betti numbers are equal. This duality actually found its modern formulation only 40

years later, with the formulation of the notions of cohomology, cup and cap products. Alexander duality refers to a conjecture formulated by J.W. Alexander in 1915 and further developed by P. Alexandrov and L. Pontryagin, relating the Betti numbers of an object (actually, a compact, locally contractible subspace of the n-sphere and those of its complement in this n-sphere). Formally, Alexander duality states that if K a compact, locally contractible subspace of the n-sphere $\mathbb{S}^n$, then,

$$\text{for all } q \geq 0 \quad \tilde{H}_q(K) \approx \tilde{H}^{n-q-1}(\mathbb{S}^n \backslash K)$$

This means that there is a relation between the holes of an object and the holes of its complement space in a sphere.

In parallel, since the 80's, with the development of computational homology, the question of constructivism has become an important matter, see for example the works on effective homology in RUBIO et al. 1986. However, to the best of our knowledge, the only works related to the constructiveness of Alexander duality are those of W. Julian, R. Mines and F. Richman (JULIAN et al. 1983) where Alexander duality theorem is developed in a constructive way in the sense of Bishop, yielding a constructive Jordan-Brouwer theorem. However, as pointed out at the very end of the article, computing generators of the complement may fail with the polyhedral approximation developed in the paper. Moreover, the complexity of the approach makes it technically unusable.

In Topological Data Analysis, Alexander duality is often used to compute homology of dimension 0 in the complementary of an n-dimensional simplicial complex instead of its $(n-1)$-dimensional homology (see, for instance, DELFINADO et al. 1995; KURLIN 2014; MAHLER et al. 2018; OBAYASHI 2017) because 0-dimensional homology comes to a simple graph problem. For instance, 0-dimensional persistent homology of a filtration can be computed in $O(n \log n)$ time using a union-find data structure, which is much faster than the algorithms for other dimensions.

This chapter firstly investigates the links between the homology of a complex and its complement, and introduces a combinatorial proof of Alexander duality for cellular and geometric complexes, see Section 4.2. Through the lens of HDVFs, the proofs of these results are constructive. More generally, the formalism of this section shows that HDVFs allow us to explicitly build the morphisms involved in standard homology results.

Next section introduces the concept of *homology configuration* (Section 4.3). Homology configurations were initially defined in an attempt to study the "constructiveness" of Alexander duality isomorphisms. However, the properties of homology configurations detached them from this original purpose, and established them as an object of study in their own right. Indeed, homology configurations provide a higher point of view on the computations of homology, partitioning HDVFs and explicit bases. It allows a deeper understanding of the entanglement between homology and cohomology computations, and provides a low dimensional topological invariant.

The general question of whether homology isomorphisms are constructible by

HDVF, is investigated later in Section 4.4, through the lens of homology configurations.

The results of Section 4.2 have partially been published in the article *A Constructive Approach to Alexander Duality*, see GONZALEZ-LORENZO, BAC, and GAZULL 2024. Note that BJÖRNER et al. 2008 also studied a combinatorial version of Alexander duality. However, their work uses a different combinatorial context, and focuses on other objects, notably the *Alexander dual* of a simplicial complex.

4.2. A Constructive Approach to Alexander Duality

The section is divided in several parts. The first part is a short introduction to reduced and relative homology. We then investigate the combination of a complex and its complementary through the lens of HDVFs, which sets up the formalism for next parts.

In the following part, entitled the *combinatorial duality*, we establish that $H_q(K) \approx H_{q+1}(L\backslash K)$, where L is an *acyclic* complex and K is a subcomplex of L closed under the boundary operator. The proof of this result is based on combinatorial and algebraic properties of HDVFs and combinatorial chain complexes. The proof also provides an explicit isomorphism that can be computed in polynomial time.

Then, the next part shows how this can be applied to recover a *combinatorial and constructive version of Alexander duality for geometric complexes*. Precisely, we show how to build an isomorphism $\tilde{H}^{n-q-1}((S\backslash K)^*) \to \tilde{H}_q(K)$, with S a geometric complex whose realization is a n-sphere.

The last part, called the *singular Alexander duality for simplicial complexes*, extends the previous result to singular homology of simplicial complexes, using subdivisions, deformation retracts and HDVFs. Specifically, we prove that $\tilde{H}_q(|K|) \approx \tilde{H}^{n-q-1}(|S|\backslash|K|)$, where S is a triangulated n-sphere.

In this section, $\mathscr{R}$ is defined to be a **field**. Though, the results of Sections 4.2.1 and 4.2.2 are also valid if $\mathscr{R}$ is a ring, but some of them are subject to the existence of perfect HDVFs. In Section 4.2.3, only the final Theorem 4.2.6 may fail, as its proof requires a perfect HDVF completion, which is not guaranteed when $\mathscr{R}$ is a ring.

4.2.1. Relative and reduced homology

Relative Homology

Given two chain complexes $(K,\partial) \subseteq (L,\partial)$, their *relative complex* $((L,K),\partial^{(L,K)})$ is the following sequence of quotient spaces L_q/K_q, with the boundary map $\partial^{(L,K)}$ naturally induced by the quotient.

$$\cdots \xrightarrow{\partial^{(L,K)}_{q+1}} L_q/K_q \xrightarrow{\partial^{(L,K)}_{q}} L_{q-1}/K_{q-1} \xrightarrow{\partial^{(L,K)}_{q-1}} \cdots$$

This relative complex is a chain complex if and only if $\partial^{(L,K)} \circ \partial^{(L,K)} = 0$. In this case, the homology of the relative complex is called the *relative homology* of the pair (L,K)

and is denoted by $H_q(L,K)$.

In general, $((L,K),\partial^{(L,K)})$ is not necessarily a chain complex, see for instance Figure 4.1. However, when L and K are geometric complexes, $((L,K),\partial^{(L,K)})$ is always a proper chain complex and their relative homology is well-defined. This is also the case in general when K is a subcomplex closed under ∂. Relative homology is also well-defined when L and K are singular chain complexes.

For combinatorial chain complexes, considering $((L,K),\partial^{(L,K)})$ is equivalent to considering the combinatorial chain subcomplex generated by the cells of L that are not in K, and the boundary map restricted to it. Precisely, we denote $(L\backslash K,\partial)$ the combinatorial chain subcomplex of (L,∂) induced by the cells $\overline{L\backslash K} := \overline{L}\backslash\overline{K}$, see Figure 4.2 for an illustration. Then, $(L\backslash K,\partial)$ is a proper chain complex whenever $((L,K),\partial^{(L,K)})$ is one, and vice-versa. In this case, there exists a chain isomorphism between $((L,K),\partial^{(L,K)})$ and $(L\backslash K,\partial)$.

Thus, $((L,K),\partial^{(L,K)})$ can be considered as the combinatorial chain subcomplex $(L\backslash K,\partial)$ of L. We adopt this later viewpoint, as it allows us to consider the cells themselves instead of elements of the quotients L_q/K_q. Note that both point of views are valid, and the article GONZALEZ-LORENZO, BAC, and GAZULL 2024 uses the other point of view and defines HDVFs for relative homology.

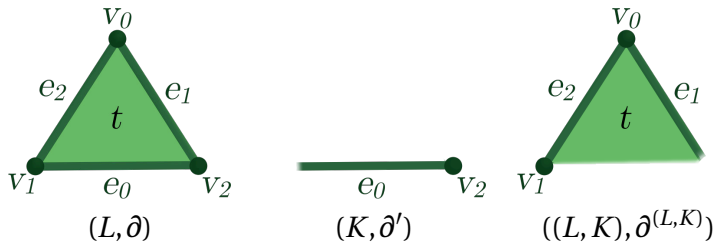


Figure 4.1 – Two complexes $(K,\partial') \subseteq (L,\partial)$ that do not define a proper relative complex. Indeed, $((L,K),\partial^{(L,K)})$ is not a chain complex, as $\partial^{(L,K)}\partial^{(L,K)}(t) = \partial^{(L,K)}(e_1+e_2) = v_1 \neq 0$ (using coefficients in $\mathbb{Z}/2\mathbb{Z}$). In addition, (K,∂') does not define a proper augmented chain complex, as $\epsilon\partial'(e_0) = \epsilon(v_2) = 1 \neq 0$.

Reduced Homology

Let (K,∂) be a chain complex, its *augmented chain complex*, $\tilde{K}$, is the chain complex K except that ∂_0 is replaced by $\epsilon : K_0 \to \mathscr{R}$, $\epsilon(x) = \sum_{\sigma\in\overline{K_0}}\langle x,\sigma\rangle$.

$$\cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\epsilon} \mathscr{R} \xrightarrow{0} \{0\}$$

The augmented complex $\tilde{K}$ is a chain complex if and only if $\epsilon\circ\partial_1 = 0$. In this case, $\tilde{K}$ defines the *reduced homology groups* $\tilde{H}_q(K) := H_q(\tilde{K})$. They are related to the ordinary homology groups by the following relations:

$$H_0(K) \approx \tilde{H}_0(K) \oplus \mathscr{R}$$
$$H_q(K) = \tilde{H}_q(K) \qquad q \geq 1$$

Reduced cohomology can be defined similarly by $\tilde{H}^q(K) := H^q(\tilde{K})$.

In general, $\tilde{K}$ is not necessarily a chain complex, see for instance Figure 4.1. However, every geometric complex and singular chain complex satisfies $\epsilon \circ \partial_1 = 0$.

Relative homology also generalizes reduced homology. Indeed, assuming that (K, ∂) is a combinatorial chain complex satisfying $\epsilon \circ \partial_1 = 0$, we obtain the following isomorphisms for any 0-cell τ_0 of K:

$$\tilde{H}_q(K) \approx H_q(K, \mathrm{Span}\{\tau_0\}) \approx H_q(K \backslash \tau_0)$$

where $K\backslash\tau_0 := \mathrm{Span}(\overline{K}\backslash\{\tau_0\})$ is the complex K without the 0-cell τ_0. See Figure 4.2 for an illustration. The explicit isomorphisms are given by:

$$\begin{aligned} \tilde{\iota}_q : \tilde{H}_q(K) &\to H_q(K\backslash\tau_0) & (\tilde{\iota}_q)^{-1} : H_q(K\backslash\tau_0) &\to \tilde{H}_q(K) \\ [x]_q &\mapsto [\mathrm{j}_{K\backslash\tau_0}(x)] = [x - \langle x, \tau_0\rangle \tau_0] & [x]_q &\mapsto [x - \epsilon(x)\tau_0] \end{aligned}$$

See Hatcher 2002 for more details about reduced and relative homology.

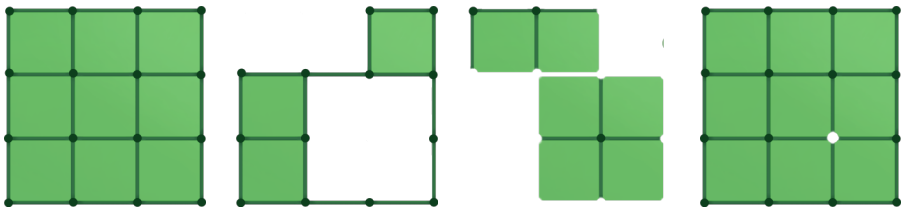


Figure 4.2 – Illustration of relative and reduced homology of combinatorial chain complexes. (K, ∂) is a subcomplex of (L, ∂). Both (K, ∂) and (L, ∂) are geometric complexes (cubical complexes) so their relative complex $((L, K), \partial^{(L,K)})$ is well-defined and isomorphic to the complex $(L\backslash K, \partial)$. In particular, these three complexes satisfy the setting of Section 4.2.2, with $K^1 = K$ and $K^0 = L\backslash K$. Moreover, the augmented chain complex of (L, ∂) is well-defined, and its homology is $\tilde{H}_q(K)$ and is the same as the homology of the complex $(L\backslash\tau_0, \partial)$.

4.2.2. Combining complexes and HDVFs

From here, we consider combinatorial chain complexes, which we refer as *complexes*. Later in this section, we will consider more restricted complexes such as *geometric* complexes and *simplicial* complexes.

This subsection introduces results that were not part of Gonzalez-Lorenzo, Bac, and Gazull 2024 and bring a better understanding on both combinatorial duality and Alexander duality. It will also be central in Section 4.4. We first investigate how to combine two disjoint complexes to form a larger complex. We then investigate this combination under the lens of two perfect HDVFs computed on these two disjoint complexes.

Combining two complementary complexes

Let (L,∂) be a complex, split into two disjoint subcomplexes (K^0,∂^0) and (K^1,∂^1). Formaly, (L,∂), (K^0,∂^0) and (K^1,∂^1) satisfy the following properties, and we write $(L,\partial) = (K^0,\partial^0) \oplus (K^1,\partial^1)$:

- $\overline{K^0} \cup \overline{K^1} = \overline{K}$ (so $K^0 \oplus K^1 = L$).
- $\overline{K^0} \cap \overline{K^1} = \emptyset$ (so $K^0 \cap K^1 = \{0\}$).
- $\partial^0 = \mathtt{j}_{K^0} \circ \partial \circ \mathtt{i}_{K^0}$ and $\partial^0 \circ \partial^0 = 0$.
- $\partial^1 = \mathtt{j}_{K^1} \circ \partial \circ \mathtt{i}_{K^1}$ and $\partial^1 \circ \partial^1 = 0$.

Remark. We have $(K^1,\partial) = (L \backslash K^0,\partial)$. Using the relative homology terminology, (K^1,∂^1) can also be considered as the relative complex $((L,K^0),\partial)$.

Note that $\mathtt{j}_{K^0} \circ \partial \circ \mathtt{i}_{K^1}$ and $\mathtt{j}_{K^1} \circ \partial \circ \mathtt{i}_{K^0}$ might be non-zero. See Figure 4.2 for an illustration of three complexes satisfying this setting.

Denote $[\cdot]^0_q : \ker\partial^0_q \to H_q(K^0)$ that maps a q-cycle to its q-homology class, and $s^0_q : H_q(K^0) \to \ker\partial^0_q$ a corresponding section, i.e. a map that associates to each homology class a representative q-cycle. In particular, $[\cdot]^0_q \circ s^0_q = id$ and for all $x \in \ker\partial^0_q$ there exists $y \in K^0_{q+1}$ such that $s^0_q \circ [\cdot]^0_q(x) = x + \partial(y)$.

Similarly, we define the maps $[\cdot]^1_q : \ker\partial^1_q \to H_q(K^1)$ and $s^1_q : H_q(K^1) \to \ker\partial^1_q$ for the complex K^1.

In what follows, we omit the dimension subscripts to improve clarity. We write ∂^1_0 and ∂^0_1 instead of $\mathtt{j}_{K^1}\partial\mathtt{i}_{K^0}$ and $\mathtt{j}_{K^0}\partial\mathtt{i}_{K^1}$ respectively. Recall that we also write $\partial^0 = \mathtt{j}_{K^0}\partial\mathtt{i}_{K^0}$ and $\partial^1 = \mathtt{j}_{K^1}\partial\mathtt{i}_{K^1}$.

Lemma 4.2.1. *We have the following equalities:*

1. $\partial^0_1\partial^1_0 = 0$ 2. $\partial^1_0\partial^0_1 = 0$ 3. $\partial^0\partial^1_0 + \partial^0_1\partial^1 = 0$ 4. $\partial^1_0\partial^0 + \partial^1\partial^1_0 = 0$

Proof. As $L = K^0 \oplus K^1$, we can write:

$$\partial = \begin{pmatrix} \partial^0 & \partial^0_1 \\ \partial^1_0 & \partial^1 \end{pmatrix}$$

This implies the following:

$$\partial\partial = 0 = \begin{pmatrix} \partial^0\partial^0 + \partial^0_1\partial^1_0 & \partial^0\partial^1_0 + \partial^0_1\partial^1 \\ \partial^1_0\partial^0 + \partial^1\partial^1_0 & \partial^1_0\partial^0_1 + \partial^1\partial^1 \end{pmatrix}$$

As we have $\partial^0\partial^0 = 0$ and $\partial^1\partial^1 = 0$, we obtain all four equalities. □

For every dimension q we can define a map α^1_0 linking the $(q+1)$-homology group of K^0 to the q-homology group of K^1.

$$\begin{aligned} \alpha^1_0 : H_{q+1}(K^0) &\to H_q(K^1) \\ h &\mapsto [\cdot]^1_q\partial^1_0 s^0_{q+1}(h) \end{aligned}$$

This map is well-defined, indeed, we have $\mathrm{im}(\partial_0^1 s^0) \subseteq \ker \partial^1$ because $\partial^1 \partial_0^1 s^0 = -\partial_0^1 \partial^0 s^0 = 0$ (by Lemma 4.2.1(4) and because $\partial^0 s^0 = 0$).

Remark. The map α_0^1 is usually defined implicitly with relative homology for geometric complexes, using the fact that $H_{q+1}(K, K^1) \approx H_{q+1}(K^0)$ and the following long exact sequence:

$$\ldots \longrightarrow H_{q+1}(K^1) \longrightarrow H_{q+1}(K) \longrightarrow H_{q+1}(K, K^1) \longrightarrow H_q(K^1) \longrightarrow H_q(K) \longrightarrow \ldots$$

For more details about these ideas, see HATCHER 2002(section 2.1).

Combining two perfect HDVFs

Let $X^0 = (P^0, S^0, C^0)$ and $X^1 = (P^1, S^1, C^1)$ be perfect HDVFs for K^0 and K^1 respectively, with reductions (f^0, g^0, h^0) and (f^1, g^1, h^1).

Denote $\mathrm{Hom}(G, G')$ the set of morphisms from a group G to a group G'. Using Lemma 1.6.2, X^0 and X^1 naturally induce the following isomorphism, whose associated commutative diagram is represented in Figure 4.3:

$$\begin{aligned} \alpha_{X^0}^{X^1} : \mathrm{Hom}\left(\mathrm{Span}\, C_{q+1}^0,\ \mathrm{Span}\, C_q^1\right) &\to \mathrm{Hom}\left(H_{q+1}(K^0),\ H_q(K^1)\right) \\ D &\mapsto [\cdot]^1 g^1 \circ D \circ f^0 s^0 \end{aligned}$$

Precisely, $\alpha_{X^0}^{X^1}$ transforms every map from the critical cells of X^0 to those of X^1, into a map from the homology of K^0 to the homology of K^1, using the reduction maps of X^0 and X^1. In parallel, X^0 and X^1 define the following map

$$\begin{aligned} D_{X^0}^{X^1} : \mathrm{Span}\, C_{q+1}^0 &\to \mathrm{Span}\, C_q^1 \\ c &\mapsto f^1 \circ \partial_0^1 \circ g^0(c) \end{aligned}$$

Intuitively, this map takes the homology generator of a critical cell in K^0, looks at its boundary (which is a cycle in K^1), and returns the critical cells in K^1 associated to it. Note that $D_{X^0}^{X^1}$ is well-defined even if X^0 and X^1 are not perfect, even though the intuition behind it is harder to understand.

It turns out that for every pair (X^0, X^1), the image of $D_{X^0}^{X^1}$ under $\alpha_{X^0}^{X^1}$ is equal to the map α_0^1, and thus does not depend on the choice of the HDVFs X^0 and X^1. This is stated in the following proposition:

Proposition 4.2.2. *Given X^0 and X^1 two perfect HDVFs for K^0 and K^1, respectively, we have $\alpha_{X^0}^{X^1}\left(D_{X^0}^{X^1}\right) = \alpha_0^1$. See Figure 4.3 for an illustration.*

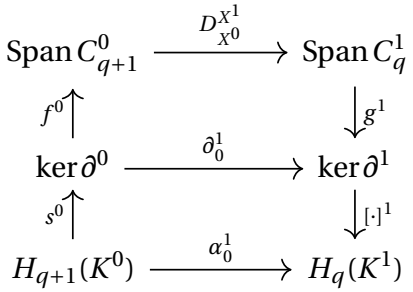


Figure 4.3 – A commutative diagram illustrating α^1_0, $\alpha^{X^1}_{X^0}$, $D^{X^1}_{X^0}$ and Proposition 4.2.2

Proof.

$$\begin{aligned}
\alpha^{X^1}_{X^0}\left(D^{X^1}_{X^0}\right) &= [\cdot]^1 g^1 f^1 \partial^1_0 g^0 f^0 s^0 \\
&= [\cdot]^1 g^1 f^1 \partial^1_0 (id - \partial^0 h^0 - h^0 \partial^0) s^0 && \text{by Definition 1.4.2} \\
&= [\cdot]^1 g^1 (f^1 \partial^1_0 s^0 - f^1 \partial^1_0 \partial^0 h^0 s^0 - f^1 \partial^1_0 h^0 \partial^0 s^0) \\
&= [\cdot]^1 g^1 (f^1 \partial^1_0 s^0 - f^1 \partial^1_0 \partial^0 h^0 s^0) && \text{because } \partial^0 s^0 = 0 \\
&= [\cdot]^1 g^1 (f^1 \partial^1_0 s^0 + f^1 \partial^1 \partial^1_0 h^0 s^0) && \text{by Lemma 4.2.1(4)} \\
&= [\cdot]^1 g^1 f^1 \partial^1_0 s^0 && \text{because } f^1 \partial^1 = 0 \\
&= [\cdot]^1 (id - \partial^1 h^1 - h^1 \partial^1) \partial^1_0 s^0 && \text{by Definition 1.4.2} \\
&= [\cdot]^1 \partial^1_0 s^0 - [\cdot]^1 \partial^1 h^1 \partial^1_0 s^0 - [\cdot]^1 h^1 \partial^1 \partial^1_0 s^0 \\
&= \alpha^1_0 - [\cdot]^1 h^1 \partial^1 \partial^1_0 s^0 && \text{because } [\cdot]^1 \partial^1 = 0 \\
&= \alpha^1_0 + [\cdot]^1 h^1 \partial^1_0 \partial_0 s^0 && \text{by Lemma 4.2.1(4)} \\
&= \alpha^1_0 && \text{because } \partial^0 s^0 = 0 \ \square
\end{aligned}$$

Proposition 4.2.2 proves that combining two perfect HDVFs on two complementary complexes yields the standard morphism between their homology groups. Using the relative homology terminology, a perfect HDVF for (K,∂), together with a perfect HDVF for $((L,K),\partial)$, induce the morphism $H_{q+1}(L,K) \to H_q(K)$ defined in HATCHER 2002(Theorem 2.16).

Note that we can do the same constructions for cohomology. To do that, we define the maps between cocycles and cohomology groups of K^0 and K^1: $[\cdot]^{*0} : \ker\partial^{0*} \to H^q(K^0)$, $[\cdot]^{*1} : \ker\partial^{1*} \to H^q(K^1)$, $s^{*0} : H^q(K^0) \to \ker\partial^{0*}$ and $s^{*1} : H^q(K^1) \to \ker\partial^{1*}$.

Then the map $\alpha^{*0}_1 := [\cdot]^{*0} \circ \left(\partial^1_0\right)^* \circ s^{*1}$ is a morphism from $H^q(K^1)$ to $H^{q+1}(K^0)$.

In addition, given two perfect HDVFs X^0 and X^1 for K^0 and K^1, with reductions (f^0,g^0,h^0) and (f^1,g^1,h^1), respectively, we have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Span} C^0_{q+1} & \xleftarrow{\left(D^{X^1}_{X^0}\right)^* = g^{0*}\partial_0^{1*}f^{1*}} & \operatorname{Span} C^1_q \\ {\scriptstyle [\cdot]^{*0} f^{0*}}\downarrow & & \uparrow{\scriptstyle g^{1*}s^{*1}} \\ H_{q+1}(K^0) & \xleftarrow{\alpha^{*0}_1} & H_q(K^1) \end{array}$$

4.2.3. The combinatorial duality

The aim of this subsection is to prove a homology relation, called *combinatorial duality* (see Corollary 4.2.7), between a complex and its complementary in a bigger *acyclic* complex. A complex (L,∂) is said to be *acyclic* if its homology is trivial, i.e. if $H_q(L) = \{0\}$ for all q. See Figure 4.4 for examples of acyclic complexes.

Geometric complexes cannot be acyclic, as they always have a non trivial 0-homology. However, we can easily build acyclic complex as combinatorial subcomplex of simple geometric complex.

For instance, let (D,∂) be a geometric complex whose realization is contractible (i.e. it is homotopy equivalent to a point). Then, removing a 0-cell from D yields an acyclic complex. Indeed, the homology of this subcomplex is isomorphic to the reduced homology of D, as pointed out in Section 4.2.1, which is zero for all dimensions.

Similarly, removing a n-cell and a 0-cell from a spherical geometric complex yields an acyclic complex. We write this statement formally in the following lemma.

Lemma 4.2.3. *Let (S,∂) be a geometric complex whose realization is homeomorphic to a n-sphere. Denote $\overline{L} := \overline{S}\backslash\{\tau_0,\tau_n\}$, with τ_0 and τ_n arbitrary cells of dimension 0 and n respectively. Denote $L = \operatorname{Span}\overline{L}$.*

Then, (L,∂^L_L) is a combinatorial subcomplex of (S,∂) that is acyclic. See Figure 4.4 for an illustration of L.

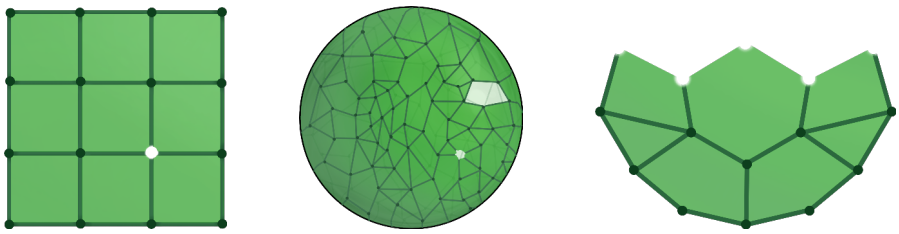


Figure 4.4 – Examples of three acyclic complexes. The first one is the geometric complex of a contractible space with a 0-cell removed. The second is a sphere with a n-cell and a 0-cell removed, as in Lemma 4.2.3. The last one is more complicated.

Proof. As τ_n is a cell of maximal dimension in S, removing it from S yields a geometric subcomplex (D,∂^D_D), with $D = S\backslash\tau_n$. The geometric realization of D is contractible, as it corresponds to a n-sphere punctured by a n-disk.

Removing τ_0 from D yields the combinatorial subcomplex (L,∂^L_L), which is well-defined as τ_0 is of minimal dimension so it cannot break the $\partial\circ\partial = 0$ assumption.

As D is a geometric complex, its reduced homology is well-defined. As pointed out in Section 4.2.1, its reduced homology is isomorphic to the homology of $D \setminus \tau_0 = L$. The reduced q-homology of D is $\{0\}$, as it is contractible, so for all q, $H_q(L) \approx \{0\}$. □

In this section we consider a n-dimensional acyclic complex (L, ∂). As in the previous subsection, we consider three complexes (L, ∂), (K^0, ∂^0) and (K^1, ∂^1) such that $(L, \partial) = (K^0, \partial^0) \oplus (K^1, \partial^1)$.

In addition, we now assume that (K^1, ∂^1) is closed under ∂ (see Definition 1.3.8). As shown by the following proposition, this assumption is equivalent to (K^0, ∂^0) being closed under ∂^*, and to $\partial^0_1 = 0$. See Figure 4.5 for an illustration of a triplet of such complexes.

Proposition 4.2.4. *If $(L, \partial) = (K^0, \partial^0) \oplus (K^1, \partial^1)$, the following properties are equivalent:*

1. *(K^1, ∂^1) is closed under ∂.*
2. *(K^0, ∂^0) is closed under ∂^*.*
3. *$\partial^0_1 = 0$*

Proof. As $L = K^0 \oplus K^1$, we have:

$$\mathtt{i}_L = \mathtt{i}_{K^0} + \mathtt{i}_{K^1} \qquad \mathtt{j}_L = \mathtt{j}_{K^0} + \mathtt{j}_{K^1} \tag{4.1}$$

Thus we obtain the following equalities:

$$\begin{aligned}
\partial \circ \mathtt{i}_{K^1} - \mathtt{j}_{K^1} \circ \partial \circ \mathtt{i}_{K^1} &= (\mathtt{j}_L - \mathtt{j}_{K^1}) \partial \mathtt{i}_{K^1} \\
&= \mathtt{j}_{K^0} \partial \mathtt{i}_{K^1} = \partial^0_1 && \text{by Equation (4.1)} \\
&= \mathtt{j}_{K^0} \partial (\mathtt{i}_L - \mathtt{i}_{K^0}) && \text{by Equation (4.1)} \\
&= \mathtt{j}_{K^0} \partial - \mathtt{j}_{K^0} \partial \mathtt{i}_{K^0} \\
&= (\partial^* \circ \mathtt{i}_{K^0} - \mathtt{j}_{K^0} \circ \partial^* \circ \mathtt{i}_{K^0})^*
\end{aligned}$$

Thus, $\partial \circ \mathtt{i}_{K^1} = \mathtt{j}_{K^1} \circ \partial \circ \mathtt{i}_{K^1}$ is equivalent to $\partial^0_1 = 0$ and to $\partial^* \circ \mathtt{i}_{K^0} = \mathtt{j}_{K^0} \circ \partial^* \circ \mathtt{i}_{K^0}$. □

To state the homological relation between K^0 and K^1, we use two perfect HDVFs X^0 and X^1 on K^0 and K^1, respectively. The main idea is to consider the reduced boundary of the HDVF $X = X^0 \sqcup X^1$ on L, and use the acyclicity of L to prove that it induces an isomorphism between the critical cells of X^0 and X^1. This result is formally stated in Theorem 4.2.6, and uses Lemmas 1.6.12 and 4.2.5.

Lemma 4.2.5. *Let $X^0 = (P^0, S^0, C^0)$ and $X^1 = (P^1, S^1, C^1)$ be HDVFs (not necessarily perfect) for K^0 and K^1 respectively. Let d^0 and d^1 be their corresponding reduced boundaries. Let $X = (P, S, C) = (P^0 \sqcup P^1, S^0 \sqcup S^1, C^0 \sqcup C^1)$ be the union of X^0 and X^1.*

Then X is a HDVF for (L,∂), and its reduced boundary d is given by:

$$d = \begin{array}{c} \\ C^0 \\ C^1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} C^0 & C^1 \end{array} \\ \begin{pmatrix} d^0 & 0 \\ D^{X^1}_{X^0} & d^1 \end{pmatrix} \end{array} \qquad \text{with} \quad D^{X^1}_{X^0} = f^1 \partial^1_0 g^0$$

Proof. First, X is a partition of the cells $\bar{L}$ because X^0 and X^1 are partitions of $\overline{K^0}$ and $\overline{K^1}$, and $\bar{L} = \overline{K^0} \sqcup \overline{K^1}$. Second, as $\partial^0_1 = 0$, we also have $\partial^{P^0}_{S^1} = 0$. Then,

$$\partial^P_S = \begin{array}{c} \\ P^0 \\ P^1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} S^0 & S^1 \end{array} \\ \begin{pmatrix} \partial^{P^0}_{S^0} & 0 \\ \partial^{P^1}_{S^0} & \partial^{P^1}_{S^1} \end{pmatrix} \end{array}$$

Note that $\partial^{P^0}_{S^0}$ and $\partial^{P^1}_{S^1}$ are invertible matrices because X^0 and X^1 are HDVF. Hence, as ∂^P_S is a lower triangular block matrix with an invertible diagonal, it is also invertible, and thus, X is a HDVF. Moreover, the inverse of ∂^P_S is given by:

$$\left(\partial^P_S\right)^{-1} = \begin{pmatrix} \left(\partial^{P^0}_{S^0}\right)^{-1} & 0 \\ -\left(\partial^{P^0}_{S^0}\right)^{-1} \partial^{P^1}_{S^0} \left(\partial^{P^1}_{S^1}\right)^{-1} & \left(\partial^{P^1}_{S^1}\right)^{-1} \end{pmatrix} = \begin{pmatrix} H^0 & 0 \\ -H^1 \partial^{P^1}_{S^0} H^0 & H^1 \end{pmatrix}$$

Consequently, by Proposition 1.6.1 the matrix of the reduced boundary of X is

$$d = \begin{array}{c} \\ C^0 \\ C^1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} C^0 & C^1 \end{array} \\ \begin{pmatrix} \partial^{C^0}_{C^0} & 0 \\ \partial^{C^1}_{C^0} & \partial^{C^1}_{C^1} \end{pmatrix} \end{array} - \begin{array}{c} \\ C^0 \\ C^1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} S^0 & S^1 \end{array} \\ \begin{pmatrix} \partial^{C^0}_{S^0} & 0 \\ \partial^{C^1}_{S^0} & \partial^{C^1}_{S^1} \end{pmatrix} \end{array} \cdot \begin{array}{c} \\ S^0 \\ S^1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} P^0 & P^1 \end{array} \\ \begin{pmatrix} H^0 & 0 \\ -H^1 \partial^{P^1}_{S^0} H^0 & H^1 \end{pmatrix} \end{array} \cdot \begin{array}{c} \\ P^0 \\ P^1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} C^0 & C^1 \end{array} \\ \begin{pmatrix} \partial^{P^0}_{C^0} & 0 \\ \partial^{P^1}_{C^0} & \partial^{P^1}_{C^1} \end{pmatrix} \end{array}$$

By developing this expression, we obtain

$$d = \begin{pmatrix} d^0 & 0 \\ \partial^{C^1}_{C^0} + F^1 \partial^{P^1}_{C^0} + F^1 \partial^{P^1}_{S^0} G^0 + \partial^{C^1}_{S^0} G^0 & d^1 \end{pmatrix}$$

In addition, we have

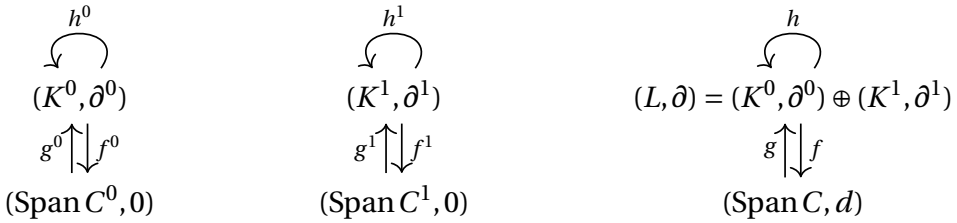
$$\begin{aligned} f^1 \circ \partial^1_0 \circ g^0 &= \begin{pmatrix} F^1 & 0 & id \end{pmatrix} \cdot \partial^1_0 \cdot \begin{pmatrix} 0 \\ G^0 \\ id \end{pmatrix} \\ &= \partial^{C^1}_{C^0} + F^1 \partial^{P^1}_{C^0} + F^1 \partial^{P^1}_{S^0} G^0 + \partial^{C^1}_{S^0} G^0 \end{aligned}$$

□

In our setting, X^0 and X^1 are perfect so $d^0 = d^1 = 0$, which implies that

$$d = \begin{array}{c} \begin{matrix} C^0 & C^1 \end{matrix} \\ \begin{pmatrix} 0 & 0 \\ D^{X^1}_{X^0} & 0 \end{pmatrix} \begin{matrix} C^0 \\ C^1 \end{matrix} \end{array}$$

The reductions are summarized in the following diagram:



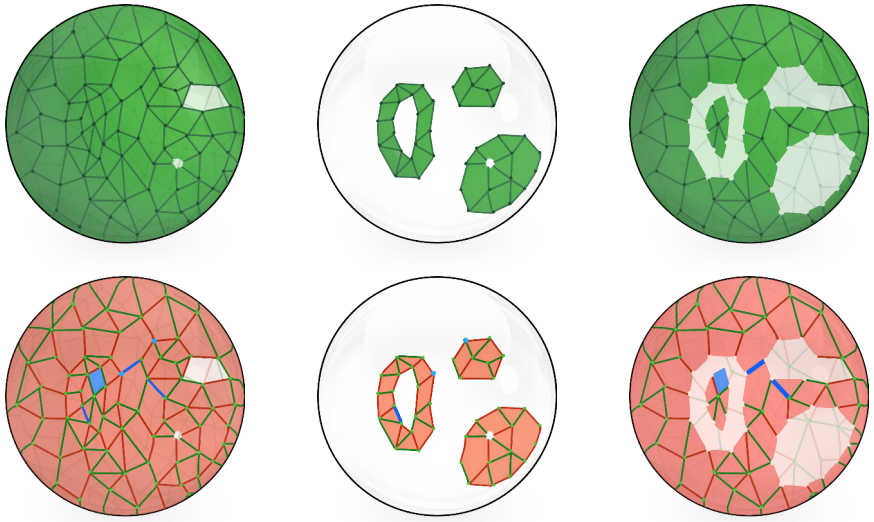


Figure 4.5 – The first row represents three complexes L, K^1 and K^0, satisfying the setting of the combinatorial duality. Indeed, we have $(L, \partial) = (K^0, \partial^0) \oplus (K^1, \partial^1)$, with L acyclic and all the properties of Proposition 4.2.4 are true. The second row represents the HDVFs X, X^1 and X^0, where X^0 and X^1 are perfect and $X = X^0 \sqcup X^1$ is the HDVF whose reduced boundary d provides $D^{X^1}_{X^0}$, see Lemma 4.2.5.

Combining the last lemma with Lemma 1.6.12 we can prove that $D^{X^1}_{X^0}$ is an isomorphism from $\operatorname{Span} C^0$ to $\operatorname{Span} C^1$.

Theorem 4.2.6 (combinatorial duality)**.** *Let $(L, \partial) = (K^0, \partial^0) \oplus (K^1, \partial^1)$ be three complexes, such that $\partial^0_1 = 0$ and (L, ∂) is acyclic. Let $X^0 = (P^0, S^0, C^0)$ and $X^1 = (P^1, S^1, C^1)$ be perfect HDVFs for K^0 and K^1 respectively. Then $D^{X^1}_{X^0}$ is an isomorphism from* $\operatorname{Span} C^0$ *to* $\operatorname{Span} C^1$.

Precisely, for every dimension q, $f^1_q \circ \partial^1_0 \circ g^0_{q+1}$ is an isomorphism from C^0_{q+1} to C^1_q.

Proof. By Lemma 4.2.5, $X = (P,S,C) = (P^0 \sqcup P^1, S^0 \sqcup S^1, C^0 \sqcup C^1)$ is a HDVF for (L,∂), and its reduced boundary d is:

$$d = \begin{array}{c} \begin{matrix} C^0 & C^1 \end{matrix} \\ \begin{pmatrix} 0 & 0 \\ D^{X^1}_{X^0} & 0 \end{pmatrix} \end{array} \begin{matrix} C^0 \\ C^1 \end{matrix} \qquad \text{with } D^{X^1}_{X^0} = f^1 \partial^1_0 g^0$$

We complete X to a perfect HDVF $X' = (P \sqcup \Pi, S \sqcup \Sigma, \emptyset) = \mathtt{A}^{\Sigma}_{\Pi} X$ for (L,∂). Note that X' doesn't have any critical cells because L is acyclic, thus we have also $C = \Pi \sqcup \Sigma$.

Then, Lemma 4.2.5 implies that $\Sigma \subseteq C^0$. Indeed, if $\tau \in \Sigma \cap C^1$, we would have that $\mathtt{j}_\Pi d \mathtt{i}_\Sigma(\tau) = 0$, contradicting Lemma 1.6.12. Similarly, $\Pi \subseteq C^1$ because $\tau \in \Pi \cap C^0$ would also contradicts Lemma 1.6.12.

Because $C = \Pi \sqcup \Sigma = C^0 \sqcup C^1$, we obtain that $\Sigma = C^0$ and $\Pi = C^1$. As a consequence, Lemma 1.6.12 implies that $D^{X^1}_{X^0} = \mathtt{j}_\Pi d \mathtt{i}_\Sigma$ is invertible. □

Using the map $\alpha^{X^1}_{X^0}$ and Proposition 4.2.2, we obtain that $\alpha^1_0 = \alpha^{X^1}_{X^0}\left(D^{X^1}_{X^0}\right)$ is an isomorphism from the homology of K^0 to the homology of K^1. This is formally stated in the following corollary:

Corollary 4.2.7. *Let $(L,\partial) = (K^0,\partial^0) \oplus (K^1,\partial^1)$ be three complexes, such that $\partial^0_1 = 0$ and (L,∂) is acyclic. Then, for every dimension q, $\alpha^1_0 = [\cdot]^1_q \circ \partial^1_0 \circ s^0_{q+1}$ is an isomorphism from $H_{q+1}(K^0)$ to $H_q(K^1)$.*

Hence, in this combinatorial setting, the homology of a subcomplex (closed under ∂) of an acyclic complex is directly related to the homology of its complementary complex. In next subsections, we show how this result can be applied to recover Alexander duality, and to construct Alexander isomorphisms.

Remark. It is worth noting that $\alpha^{X^1}_{X^0}$ takes an invertible matrix D and gives the corresponding homology isomorphism. Moreover, when we complete the HDVF $X = X^0 \sqcup X^1$, we obtain different invertible matrices D – which are actually obtained by performing Gaussian eliminations on $D^{X^1}_{X^0}$. Hence, given X^0 and X^1, we can build a family of homology isomorphisms of the form $\alpha^{X^1}_{X^0}(D)$. It turns out not all homology isomorphisms can be obtained using this method. These ideas and constructions will be studied more deeply in Section 4.4, where we investigate the isomorphisms that are constructible by this principle.

4.2.4. Combinatorial Alexander duality for geometric complexes

We now adapt the previous results to the case of Alexander duality. Precisely, we use the combinatorial duality (Corollary 4.2.7) to obtain a version of Alexander duality for geometric complexes, see Theorem 4.2.8.

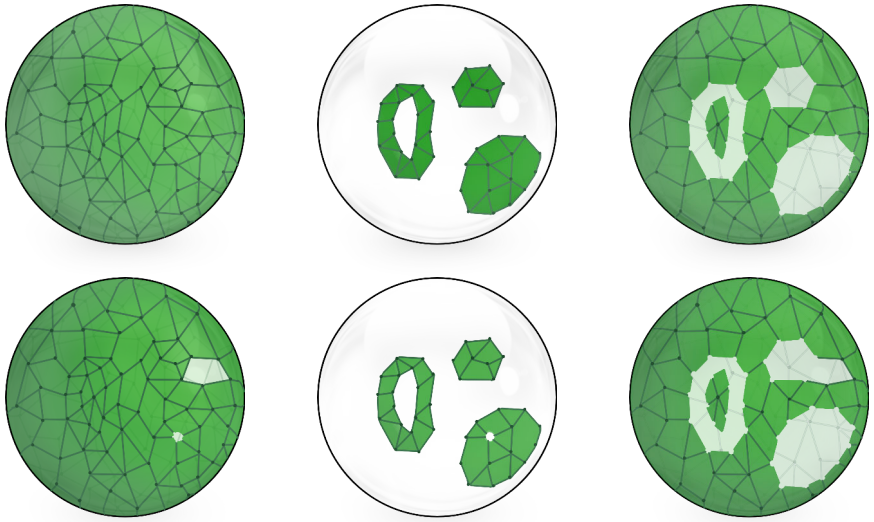


Figure 4.6 – Representation of the complexes involved in the proof of Theorem 4.2.8. The first row represents S, K and K^c. The second row represents $L = S\backslash\{\tau_0, \tau_n\}$, $K^1 = K\backslash\tau_0$ and $K^0 = K^c\backslash\tau_n$.

We consider (S,∂), a geometric complex whose realization is homeomorphic to a n-sphere. Let (K,∂) be a geometric subcomplex[1] of S. Let (K^c,∂) be its complementary subcomplex in S, i.e. $\overline{K^c} = \overline{S}\backslash\overline{K}$. In particular, $(S,\partial) = (K,\partial) \oplus (K^c,\partial)$, (K,∂) is closed under ∂ and (K^c,∂) is closed under ∂^*, by Proposition 4.2.4. It is important to keep in mind that although (K,∂) is a geometric complex, (K^c,∂) is just a simple combinatorial chain subcomplex. In particular, $H_q(|K^c|)$ is not necessarily isomorphic to $H_q(K^c)$. See Figure 4.6 for an illustration of these complexes.

We avoid the simple case where (K,∂) is empty or is equal to (S,∂). Thus, let τ_0 be a 0-cell of K. Similarly, we choose a n-cell τ_n of K^c. This is possible as K^c is closed under ∂^* and is non empty.

Denote $\overline{L} = \overline{S}\backslash\{\tau_0,\tau_n\}$ and $L = \operatorname{Span}\overline{L}$. Lemma 4.2.3 states that (L,∂) is a well-defined acyclic subcomplex of (S,∂). Note that (L,∂) is not a geometric complex.

By applying Corollary 4.2.7 to $(L,\partial) = (K^c\backslash\tau_n,\partial) \oplus (K\backslash\tau_0,\partial)$, and by combining the resulting isomorphism with reduced (co)homology, we obtain the following theorem, whose proof is detailed afterwards.

Theorem 4.2.8 (Combinatorial Alexander duality for geometric complexes)**.** *Given (S,∂) a geometric complex representing a n-sphere, and (K,∂) a geometric subcomplex of S, we have:*

$$\tilde{H}^{n-q-1}\left((K^c)^*\right) \approx \tilde{H}_q(K)$$

The whole sequence of isomorphisms can be summed up in the following diagram:

[1] We recall that, when writing "(K',∂) is a subcomplex of (K,∂)", we implicitly consider the subcomplex $(K',\partial^{K'}_{K'})$, with $\partial^{K'}_{K'} = \mathtt{j}_{K'} \circ \partial \circ \mathtt{i}_{K'}$ the boundary map restrained and corestrained to K'.

$$\begin{array}{ccccc} \tilde{H}^{n-q-1}((K^c)^*) & & & & \tilde{H}_q(K) \\ \downarrow{\scriptstyle \tilde{\iota}^{n-q-1}} & & & & \uparrow{\scriptstyle (\tilde{\iota}_q)^{-1}} \\ H^{n-q-1}((K^c\backslash\tau_n)^*) & \xrightarrow{id} & H_{q+1}(K^c\backslash\tau_n) & \xrightarrow{\alpha_0^1} & H_q(K\backslash\tau_0) \end{array}$$

As mentioned for the combinatorial duality, α_0^1 can be replaced by any $\alpha_{X^0}^{X^1}(D)$, where X^0 and X^1 are perfect HDVFs for $(K^c\backslash\tau_n)$ and $(K\backslash\tau_0)$, and D is an invertible matrix of the appropriate size.

Proof. Define $(K^1,\partial^1) = (K\backslash\tau_0,\partial)$ and $(K^0,\partial^0) = (K^c\backslash\tau_n,\partial)$. K^1 (resp. K^0) is a well-defined complex, as τ_0 (resp. τ_n) is a cell of minimal (resp. maximal) dimension, and removing it do not alter the $\partial\circ\partial = 0$ assumption. Thus, $(L,\partial) = (K^0,\partial^0)\oplus(K^1,\partial^1)$ (see Figure 4.6 for an illustration). Moreover, L is acyclic and K^1 is closed in L under ∂ (because K is closed in S under ∂). As a result, we are in the framework of the combinatorial duality of previous subsection. Hence, Corollary 4.2.7 implies that α_0^1 is an isomorphism from $H_{q+1}(K^0)$ to $H_q(K^1)$. Using the duality between homology and cohomology for finite dimensional complexes (see Section 1.3.7), we get the following isomorphisms

$$H^{n-q-1}\left((K^c\backslash\tau_n)^*\right) = H_{q+1}\left(K^c\backslash\tau_n\right) = H_{q+1}(K^0) \xrightarrow{\alpha_0^1} H_q(K^1) = H_q(K\backslash\tau_0)$$

Then, using reduced homology, we get (see Section 4.2.1):

$$\tilde{H}_q(K) \xrightarrow{\iota^q} H_q(K\backslash\tau_0)$$

We can get a similar result for $(K^c)^*$, however we first need to show that its reduced homology is well-defined. Denote $\epsilon^* : x \in S_n \to \sum_{\sigma\in S_n}\langle x,\sigma\rangle \in \mathscr{R}$ the map of the augmented chain complex of $(S,\partial)^*$ (see Section 4.2.1).

S represents a manifold without boundary, so every $(n-1)$-cell σ has exactly two cofaces σ_1 and σ_2. We have $\langle\sigma,\partial(\sigma_1)\rangle + \langle\sigma,\partial(\sigma_2)\rangle = 0$, otherwise the whole manifold would have a boundary. This implies that $\epsilon^*\circ\partial^*(\sigma) = 0$, and so for all $\sigma \in \overline{S_n}$, thus, the augmented chain complex of $(S,\partial)^*$ is well-defined.

As $(K^c)^*$ is a subcomplex of $(S,\partial)^*$ closed under ∂^*, the augmented chain complex of $(K^c)^*$ is also well-defined. τ_n is a n-cell of K^c, so it is considered a 0-cell of $(K^c)^*$. Hence, the reduced homology of $(K^c)^*$ is isomorphic to the homology of $(K^c\backslash\tau_n)^*$ (see Section 4.2.1). This directly implies the following isomorphism:

$$\tilde{H}^{n-q-1}\left((K^c)^*\right) \xrightarrow{\tilde{\iota}^{n-q-1}} H^{n-q-1}\left((K^c\backslash\tau_n)^*\right)$$

All in all, we obtain the sequence of isomorphisms presented in Theorem 4.2.8. □

Remark. Although the standard Alexander duality holds for homology over a ring, Theorem 4.2.8 is only valid for homology over a field, as it relies on HDVF

perfect completion. The study of HDVF perfect completion for homology over a ring is a major challenge of this theory.

4.2.5. Singular Alexander duality for simplicial complexes

The version of Alexander duality obtained in the previous section is not totally satisfying. Indeed, it only proves a result for the homology of complexes, but the result cannot be extended to singular homology, as (K^c,∂) is not a geometric complex. Thus, the aim of this subsection is to use our formalism to get a version of Alexander duality for singular homology, more precisely for the geometric realizations of simplicial subcomplexes.

Therefore, let S be a triangulated n-sphere, i.e. (S,∂) is a simplicial complex[2] whose geometric realization $|S|$ is homeomorphic to the n-sphere. Let (K,∂) be a simplicial subcomplex of S, so its homology group $H_q(K)$ is isomorphic to the singular homology group of $|K|$, the geometric realization of K. Denote (K^c,∂) the complementary complex of (K,∂) in (S,∂). See Figure 4.7 for an illustration.



Figure 4.7 – Illustration of the complexes of the initial setting of Section 4.2.5. Contrary to (K,∂) and (S,∂), (K^c,∂) is not a proper simplicial complex.

As previously, (K^c,∂) is not a simplicial complex. Hence, $H_{q+1}(K^c)$ is not necessarily isomorphic to the singular homology group of $|K^c| = |S|\backslash|K|$, although they are related. The relation between both groups is proved in this section, which entails a constructive Alexander duality in this geometric setting. Our proof is structured in several steps:

1. We introduce a cellular subdivision $\mathscr{S}$ such that the complex $\mathscr{S}(S)\backslash\mathscr{S}(K)$ is a subdivision of K^c, whose geometric realization coincides with $|S|\backslash|K|$ (Definition 4.2.1).
2. We then defined $\mathsf{sc}(K^c)$, a subcomplex of $\mathscr{S}(S)\backslash\mathscr{S}(K)$ that has the structure of a polyhedral complex (Definition 4.2.2 and Lemma 4.2.9).
3. We prove that $|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$ has the same homotopy type as $|\mathsf{sc}(K^c)|$ (Lemma 4.2.10).
4. Next, we build a HDVF on $\mathsf{sc}(K^c)$ to prove a relation between its homology groups and the cohomology groups of K^c (Proposition 4.2.11).

[2] Recall that when writing "let (K,∂) be a simplicial complex", we actually define a simplicial complex $\overline{K}$ whose associated simplicial chain complex is (K,∂).

5. Finally, we conclude using the version of Alexander duality that was proved in the previous subsection, see Theorem 4.2.13.

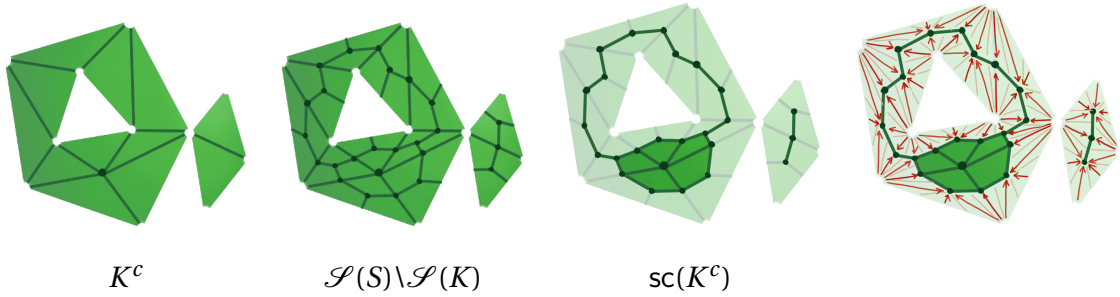


Figure 4.8 – Illustration of the steps of the proof of Theorem 4.2.13. $\mathscr{S}(S)\backslash\mathscr{S}(K)$ is the *cellular subdivision* of K^c (see Definition 4.2.1). $\mathsf{sc}(K^c)$ is the *subdivision core* of K^c (see Definition 4.2.2). The fourth image is the deformation retract from $|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$ to $|\mathsf{sc}(K^c)|$ (see Lemma 4.2.10).

See Figure 4.8 for an illustration of the constructions involved in this proof.

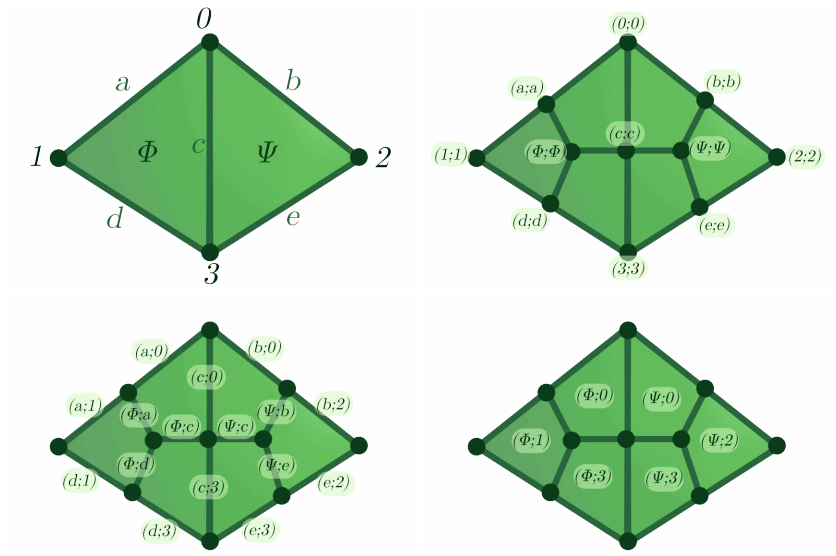


Figure 4.9 – A simplicial complex K (first image) and its cellular subdivision $\mathscr{S}(K)$ (second to fourth). The second, third and fourth images display the label of the cells $(\sigma;\tau)$ of dimension 0, 1 and 2 respectively.

Cellular subdivision and subdivision core

In this part, we prove that even though K^c is not a simplicial complex, its geometric realization $|S|\backslash|K|$ is homotopically equivalent to a polyhedral complex called the *subdivision core*. First we define a specific subdivision.

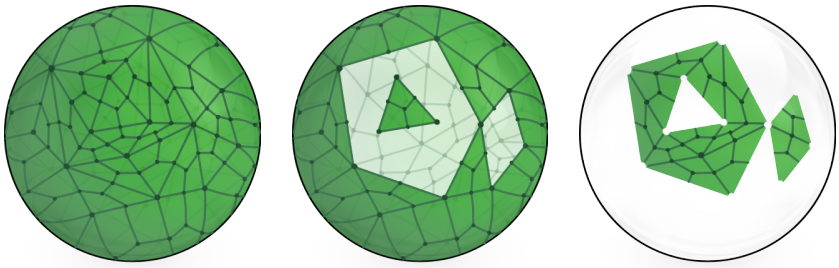
Definition 4.2.1. Let (K,∂) be a simplicial complex. Recall that given $\sigma,\tau \in \overline{K}$, $\sigma \subseteq \tau$ means that σ is a face of τ. We define the *cellular subdivision* of (K,∂) as the set $\overline{\mathscr{S}}(K) := \{(\sigma;\tau) \mid \sigma,\tau \in \overline{K}, \tau \subseteq \sigma\}$. We denote $\mathscr{S}(K) := \operatorname{Span}\overline{\mathscr{S}}(K)$, then $(\mathscr{S}(K),\partial')$ defines a polyhedral complex, where ∂' is defined as

$$\partial'(\sigma;\tau) = \sum_{\tau' \subseteq \sigma} \langle \partial(\tau'),\tau\rangle \cdot (\sigma;\tau') + \sum_{\sigma' \supseteq \tau} \langle \partial(\sigma),\sigma'\rangle \cdot (\sigma';\tau)$$

The dimension of a cell $(\sigma;\tau)$ is $\dim(\sigma;\tau) := \dim(\sigma) - \dim(\tau)$. See Figure 4.9 for an example of this subdivision.

The geometric realization of a cell $(\sigma;\tau)$ is the open convex hull of the centers of the simplices σ' such that $\tau \subseteq \sigma' \subseteq \sigma$. Precisely,

$$|(\sigma;\tau)| = \left\{ \sum_{v_i \in \sigma} \lambda_i \cdot v_i \;\middle|\; \sum \lambda_i = 1 \text{ and } \begin{pmatrix} 0 < \lambda_i < 1 & \text{for all } i \\ \lambda_i = \lambda_j & \text{if } v_i \in \tau \text{ and } v_j \in \tau \\ \lambda_i > \lambda_j & \text{if } v_i \in \tau \text{ and } v_j \notin \tau \end{pmatrix} \right\}$$



$\mathscr{S}(S)$ $\mathscr{S}(K)$ $\mathscr{S}(L)\backslash\mathscr{S}(K)$

Figure 4.10 – Illustration of the cellular subdivisions of the complexes of Figure 4.7, see Definition 4.2.1.

The geometric realization of $\mathscr{S}(K)$ is then the union of the geometric realizations of its cells. Importantly, this definition implies that $|\mathscr{S}(K)|$ is equal to $|K|$.

Figure 4.9 shows a simplicial complex and its cellular subdivision. Note how each simplex σ of the simplicial complex is decomposed into $(\sigma;\tau)$ for all $\tau \subseteq \sigma$.

We are interested in $\mathscr{S}(S)\backslash\mathscr{S}(K)$, which is also the subdivision of the simplices of K^c. $(\mathscr{S}(S)\backslash\mathscr{S}(K),\partial')$ defines a combinatorial chain subcomplex of $(\mathscr{S}(S),\partial')$ that is not a geometric subcomplex, as the geometric faces of its cells might not be in $\mathscr{S}(S)\backslash\mathscr{S}(K)$. Equivalently, it is not closed under ∂'. However, it contains a subset of cells that defines a proper geometric subcomplex, namely, the *subdivision core*.

First, let us introduce the following useful maps, defined on cells but extended linearly:

$$\begin{aligned} \vee : \mathscr{S}(S) &\to S & \qquad \wedge : \mathscr{S}(S) &\to S \\ (\sigma;\tau) &\mapsto \tau & (\sigma;\tau) &\mapsto \sigma \end{aligned}$$

Definition 4.2.2. Given $(K,\partial) \subseteq (S,\partial)$ two simplicial complexes, the *subdivision core* of (K^c,∂) is

$$\mathrm{sc}(K^c) = \mathrm{Span}\left\{(\sigma;\tau) \in \mathscr{S}(S)\backslash\mathscr{S}(K) \mid \tau \in \overline{S}\backslash\overline{K}\right\} = \vee^{-1}(K^c) \cap \wedge^{-1}(K^c)$$

See Figure 4.8 for an example of the subdivision core.

Lemma 4.2.9. *Let $(K,\partial) \subseteq (S,\partial)$ be two simplicial complexes. Then, $\mathrm{sc}(K^c)$ is a polyhedral complex. Precisely, $(\mathrm{sc}(K^c),\partial')$ is a polyhedral subcomplex of $(\mathscr{S}(S),\partial')$.*

Proof. By Proposition 1.3.2 and because polyhedral complexes forbid multi-incidence, we need to prove that the faces of each cell in $\mathrm{sc}(K^c)$ belongs to $\mathrm{sc}(K^c)$.

Let $(\sigma;\tau) \in \mathrm{sc}(K^c)$, its boundary is $\sum_{\tau'\subseteq\sigma}\langle\partial(\tau'),\tau\rangle\cdot(\sigma;\tau') + \sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot(\sigma';\tau)$. Thus, the faces of $(\sigma;\tau)$ are the $(\sigma;\tau')$ with a non-zero coefficient and the $(\sigma';\tau)$ with a non-zero coefficient.

The cells $(\sigma;\tau')$ with a non-zero coefficient are in $\mathrm{sc}(K^c)$. Indeed, $\langle\partial(\tau'),\tau\rangle \neq 1$ implies that τ is a face of τ', hence $\tau' \in K^c$ since $\tau \in K^c$, and K^c is closed under ∂^*.

On the other hand, the cells $(\sigma';\tau)$ with a non-zero coefficient are in $\mathrm{sc}(K^c)$ because $\tau \subseteq \sigma'$ and $\tau \in K^c$ imply that $\sigma' \in K^c$. □

Note that $\mathrm{sc}(K^c)$ is the largest subcomplex of $\mathscr{S}(S)\backslash\mathscr{S}(K)$ closed in $\mathscr{S}(S)$ under ∂'. The advantage of $\mathrm{sc}(K^c)$ is that its geometric realization has the same homotopy type as $|S|\backslash|K|$:

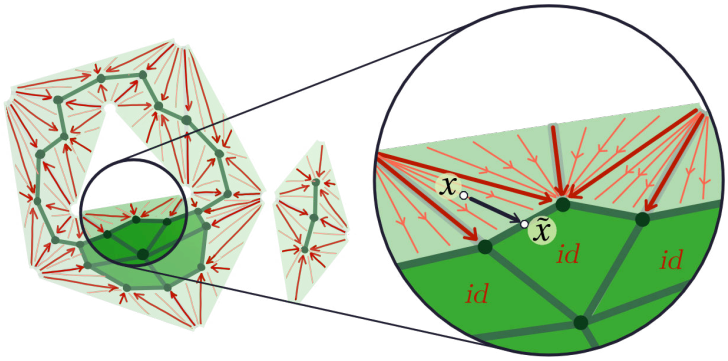


Figure 4.11 – Illustration of the deformation retract of Lemma 4.2.10. The zoom on the left highlights how each simplex of the barycentric subdivision of $\mathscr{S}(S)\backslash\mathscr{S}(K)$ is retracted to its highest dimensional face that belongs to $\mathrm{sc}(K^c)$.

Lemma 4.2.10. *Let $(K,\partial) \subseteq (S,\partial)$ be a pair of simplicial complexes. Then, $|\mathrm{sc}(K^c)|$ is a deformation retract of $|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$. See Figure 4.11 for an illustration of this retraction.*

Proof. We want to define a deformation retract from $|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$ to $|\mathsf{sc}(K^c)|$.

$$h : [0,1] \times |\mathscr{S}(S)|\backslash|\mathscr{S}(K)| \to |\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$$
$$(t,x) \mapsto h_t(x)$$

We define the $h_t(x)$ piece-wise in each cell $(\sigma;\tau) \in \mathscr{S}(S)\backslash\mathscr{S}(K)$. It is a function that maps each point in $|(\sigma;\tau)|$ to a point in $\bigcup_{\tau' \in \overline{S}\backslash\overline{K}} |(\sigma;\tau')|$.

Let $x \in |\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$. Then, x belongs to $|(\sigma;\tau)|$ for some σ and τ, so we can write x with its barycentric coordinates: $x = \sum_{v_i \in \sigma} \lambda_i \cdot v_i$. Without loss of generality, we reorder the indices of the v_i to have $\tau = [v_0, \cdots, v_j]$, $\sigma = [v_0, \cdots, v_j]$, with $j \leq q$, and $\lambda_0 = \cdots = \lambda_j > \lambda_{j+1} \geq \cdots \geq \lambda_q$.

Let m be the minimum index such that the simplex $[v_0, \ldots, v_m]$ belongs to $\overline{S}\backslash\overline{K}$, in particular $m \geq j$. To be really precise, we actually need to consider $m' = \max\{j \mid \lambda_j = \lambda_m\}$.

We denote $\tau_x := [v_0, \ldots, v_m]$ and we have $(\sigma;\tau) \in \mathsf{sc}(K^c)$. We define $h_t(x)$ using the point $\tilde{x} \in |(\sigma;\tau_x)|$ defined as follows

$$\tilde{x} = \frac{1}{\sum_{i=0}^{m} \lambda_m + \sum_{i=m+1}^{q} \lambda_i} \left(\sum_{i=0}^{m} \lambda_m \cdot v_i + \sum_{i=m+1}^{q} \lambda_i \cdot v_i \right)$$
$$h_t(x) = (1-t) \cdot x + t \cdot \tilde{x}$$

h is a deformation retract:

- $h_0 = id$;
- $h_1(x) = \tilde{x} \in |(\sigma;\tau_x)| \subseteq |\mathsf{sc}(K^c)|$;
- $\forall x \in |\mathsf{sc}(K^c)| \forall t \in [0,1], h_t(x) = x$. Indeed, if $x \in |\mathsf{sc}(K^c)|$, then $\tau = \tau_x$, so $m = j$ and we have $x = \tilde{x}$.
- The continuity of h is hard to write, we only present some arguments, without details, and ask the reader to refer to the illustration of Figure 4.11 or to the interactive online visualization[3].

  First, h is continuous on each simplex of the barycentric subdivision of K^c. Indeed, let s be a simplex of the barycentric subdivision of K^c. Two points in s produce the same ordering of λ_is, which implies that $x \mapsto \tilde{x}$ is continuous on s.

  Second, h is continuous in between s and its faces. Indeed, being on a face of s simply adds additional equalities in the ordering of λ_is. There are two cases. Either τ_x becomes a coface of itself, or σ becomes a face of itself. In both cases, this implies that $\tilde{x}$ changes from being a point of $(\sigma;\tau_x)$ to being a point of a face of $(\sigma;\tau_x)$.

□

[3] https://pageperso.lis-lab.fr/aldo.gonzalez-lorenzo/papers/alexander-duality/retraction-kernel.html

As $\mathsf{sc}(K^c)$ is a polyhedral complex, its homology groups are isomorphic to the singular homology groups of its realization. Therefore, we have for all q:

$$\begin{aligned} H_q(\mathsf{sc}(K^c)) &= H_q(|\mathsf{sc}(K^c)|) \\ &= H_q(|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|) \quad \text{as a deformation retract preserves the homology} \\ &= H_q(|S|\backslash|K|) \end{aligned}$$

The singular homology groups of the open set $|S|\backslash|K|$ and the homology groups of $(\mathsf{sc}(K^c),\partial')$ are isomorphic.

From the subdivision core to Alexander duality

In this part, we prove that we can build a HDVF on $(\mathsf{sc}(K^c),\partial')$ whose reduced boundary is strongly related to the complex (K^c,∂^*). This HDVF allows us to prove that $H_q(|S|\backslash|K|)$ is isomorphic to $H^{n-q}(K^c)$ (Proposition 4.2.11). Combined with Lemma 4.2.10 and Theorem 4.2.8, we obtain the geometric version of Alexander duality, see Theorem 4.2.13.

Proposition 4.2.11. *Let $(K,\partial)\subseteq(S,\partial)$ be a pair of simplicial complexes such that S is a triangulated n-sphere. Then, $H_q(\mathsf{sc}(K^c))\approx H^{n-q}(K^c)$.*

Precisely, the isomorphism is given by $[\cdot]_{n-q}^{K^c}\circ\vee\circ f\circ s_q^{\mathsf{sc}(K^c)}$, with f given by the reduction of a particular HDVF over $\mathsf{sc}(K^c)$.

The proof of this proposition is long and technical. The idea is to build a HDVF on $\mathsf{sc}(K^c)$ that induces a reduction to a chain complex that is isomorphic to (K^c,∂^*). See Figure 4.12 for an illustration of the proof.

First, given a cell τ, we define $\Delta^*\tau=\{\sigma,\sigma\supseteq\tau\}$ the set of all the cofaces of $\tau\in K^c$, and $\Delta_q^*\tau$ the q-cofaces of τ. $(\mathrm{Span}\,\Delta^*\tau,\partial)$ defines a subcomplex of S that is closed under ∂^*. Note that it is not a simplicial complex, see Figure 4.13 for examples. Its homology is given by the following lemma.

Lemma 4.2.12. $H_q\left(\mathrm{Span}\,\Delta^*\tau\right)\approx\{0\}$ *if* $q\neq n$ $\qquad H_n\left(\mathrm{Span}\,\Delta^*\tau\right)\approx\mathscr{R}$

Proof. $\Delta^*\tau$ is usually called the *open star* of τ in S (see Hatcher 2002(2.C)). Denote $\mathrm{St}\tau$ its *closed star* (by adding all the faces of simplices of $\Delta^*\tau$) and $\mathrm{Lk}\tau=\mathrm{St}\tau\backslash\Delta^*\tau$ its *link* in S. As S is a triangulated n-sphere, $\mathrm{St}\tau$ and $\mathrm{Lk}\tau$ are a simplicial complexes homeomorphic to a n-disk and a $(n-1)$-sphere respectively.

Let $\tau_0\in\mathrm{Lk}\tau$ be a 0-cell. As $\mathrm{St}\tau$ is contractible, $\mathrm{St}\tau\backslash\tau_0$ is acyclic. Moreover, we have $(\mathrm{Span}\,\mathrm{St}\tau\backslash\tau_0,\partial)=(\mathrm{Span}\,\mathrm{Lk}\tau\backslash\tau_0,\partial)\oplus(\mathrm{Span}\,\Delta^*\tau,\partial)$, with $(\mathrm{Span}\,\Delta^*\tau,\partial)$ closed under ∂^*. Hence, the combinatorial duality (see Theorem 4.2.6) gives for all q, $H_q(\mathrm{Span}\,\mathrm{Lk}\tau\backslash\tau_0)\approx H_{q+1}(\mathrm{Span}\,\Delta^*\tau)$.

As $H_q(\mathrm{Span}\,\mathrm{Lk}\tau\backslash\tau_0)\approx\tilde{H}_q(\mathrm{Span}\,\mathrm{Lk}\tau)$ and $\mathrm{Lk}\tau$ is homeomorphic to a $(n-1)$-sphere, we get the stated homology. □

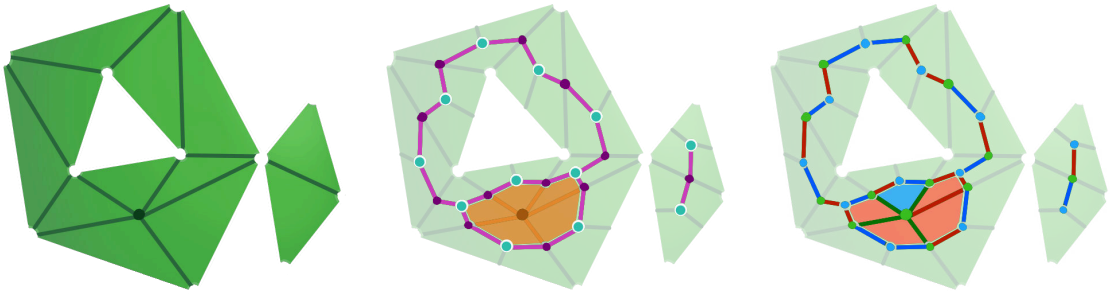


Figure 4.12 – Illustration of the objects used in the proof of Proposition 4.2.11. The first image is the complex K^c. The second image represents the subdivision core $\mathsf{sc}(K^c)$, segmented into the subcomplexes $K^{(\tau)}$. $K^{(\tau)}$ is displayed in orange, purple or cyan, whether $\dim\tau$ is equal to 0, 1 or 2, respectively. The third image is the HDVF X on $\mathsf{sc}(K^c)$, which is defined as the union of all perfect HDVFs $X^{(\tau)}$. The critical cells (in blue) are exactly the cells $(\bar{\tau};\tau)$. Together with the reduced boundary operator of X, they are isomorphic to the complex (K^c,∂^*).

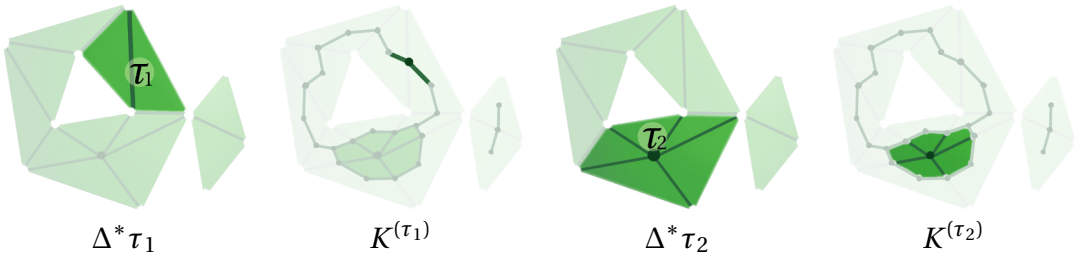


Figure 4.13 – Two examples of $\Delta^*\tau$ and their corresponding $K^{(\tau)}$, illustrating the proof of Proposition 4.2.11. The two first images consider a cell τ_1 of dimension 1, whereas the two last images consider a cell τ_2 of dimension 0.

Proof. [*Proof of Proposition 4.2.11*] We prove this isomorphism by building a HDVF for the complex $\mathsf{sc}(K^c)$.

For every cell τ of K^c, we define $K^{(\tau)} = \mathrm{Span}\{(\sigma;\tau) \in \mathsf{sc}(K^c)\}$. Alternatively, $K^{(\tau)} = \vee^{-1}(\mathrm{Span}\{\tau\}) \cap \wedge^{-1}(K^c)$. Note that every $(K^{(\tau)},\partial')$ is a subcomplex of $\mathsf{sc}(K^c)$, and that they define a partition of $\mathsf{sc}(K^c)$: $(\mathsf{sc}(K^c),\partial') = \bigoplus_{\tau\in\overline{K^c}}(K^{(\tau)},\partial')$. See Figure 4.13 for an illustration of $K^{(\tau)}$.

First, we fix a cell τ of K^c and prove that $K^{(\tau)}$ has a simple homology, see Equation (4.2). See Figure 4.12 for an illustration of the $K^{(\tau)}$.

We prove that the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & K^{(\tau)}_{q+1-\dim\tau} & \xrightarrow{\partial'} & K^{(\tau)}_{q-\dim\tau} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \wedge} & & \downarrow{\scriptstyle \wedge} & & \\
\cdots & \longrightarrow & \mathrm{Span}\left(\Delta^*_{q+1}\tau\right) & \xrightarrow{\partial} & \mathrm{Span}\left(\Delta^*_q\tau\right) & \longrightarrow & \cdots
\end{array}$$

Indeed, on $K^{(\tau)}$, the boundary map gives

$$\begin{aligned}\partial'^{K^{(\tau)}}_{K^{(\tau)}}(\sigma;\tau) &= \langle\partial(\tau),\tau\rangle\cdot(\sigma;\tau) + \sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot(\sigma';\tau)\\ &= \sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot(\sigma';\tau) \qquad \text{as } \langle\partial(\tau),\tau\rangle = 0\end{aligned}$$

On Span $(\Delta^*\tau)$, the boundary map gives

$$\partial^{\Delta^*\tau}_{\Delta^*\tau}(\sigma) = \mathtt{j}_{\Delta^*\tau}\partial(\sigma) = \sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot\sigma'$$

Hence, we have

$$\wedge\circ\partial'^{K^{(\tau)}}_{K^{(\tau)}}(\sigma;\tau) \;=\; \sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot\sigma' \;=\; \partial^{\Delta^*\tau}_{\Delta^*\tau}(\sigma) \;=\; \partial^{\Delta^*\tau}_{\Delta^*\tau}\circ\wedge(\sigma;\tau)$$

As a result, $\mathtt{j}_{\Delta^*\tau}\circ\wedge\circ\mathtt{i}_{K^{(\tau)}}$ is a chain map from $(K^{(\tau)},\partial')$ to (Span $(\Delta^*\tau)$, ∂). Moreover, as it associates (σ,τ) to σ (with τ fixed), it is an isomorphism. Thus, $\wedge$ induces an isomorphism from $H_{q-\dim\tau}(K^{(\tau)})$ to $H^q(\Delta^*\tau)$. Using Lemma 4.2.12 we obtain:

$$\beta_q\left(K^{(\tau)}\right) = 0 \text{ if } q\neq n-\dim\tau \qquad \beta_{n-\dim\tau}\left(K^{(\tau)}\right) = 1 \tag{4.2}$$

Let us now compute a HDVF for $\mathrm{sc}(K^c)$, by taking the union of perfect HDVFs for the disjoint subcomplexes $K^{(\tau)}$

Let $X^{(\tau)}$ be a perfect HDVF for the subcomplex $K^{(\tau)}$. $X^{(\tau)}$ has exactly one critical cell, whose dimension is $n-\dim(\tau)$. Denote this cell $(\bar\tau;\tau)$ with $\dim(\bar\tau)=n$. All the other cells $(\sigma;\tau)$ with $\dim(\sigma)=n$ are secondary.

The union of all $X^{(\tau)}$ results in a HDVF $X=(P,S,C)$ for $\mathrm{sc}(K^c)$. Indeed, as the $\overline{K^{(\tau)}}$ define a partition of $\overline{\mathrm{sc}(K^c)}$, X defines a partition of $\mathrm{sc}(K^c)$. Using a similar argument as in the proof of Lemma 4.2.5, we get that ∂'^P_S is invertible. Precisely, if $\dim\tau\geq\dim\tau'$, we have $\langle\partial(\tau'),\tau\rangle=0$. By looking at the formula of ∂', this implies $\partial'^{K^{(\tau')}}_{K^{(\tau)}}=0$. Thus, if we order the cells τ of K^c by increasing dimension, ∂'^P_S becomes a lower triangular block matrix, whose diagonal is $\partial'^{P^{(\tau)}}_{S^{(\tau)}}=0$. Hence, ∂'^P_S is invertible as $X^{(\tau)}$ are HDVFs.

By construction $C=\{(\bar\tau;\tau)\mid\tau\in\overline{K^c}\}$. Let (f,g,h) be the reduction from $(\mathrm{sc}(K^c),\partial')$ to (Span C,d) induced by X. See Figure 4.12 for an illustration of the X. We prove that the following diagram commutes:

$$\begin{array}{ccccccc}
\dots & \longrightarrow & \mathrm{Span}\,C_{n-q} & \xrightarrow{\;d\;} & \mathrm{Span}\,C_{n-q-1} & \longrightarrow & \dots\\
 & & \downarrow\vee & & \downarrow\vee & & \\
\dots & \longrightarrow & K^c_q & \xrightarrow{\;\partial^*\;} & K^c_{q+1} & \longrightarrow & \dots
\end{array}$$

Indeed, first, consider the reduced boundary d. By Proposition 1.6.1 we have

$d = \mathtt{j}_C \partial' g$. Moreover,

$$g(\bar{\tau};\tau) = \sum_{\sigma\in I}(\sigma;\tau), \quad I = \{\sigma \in K^c \mid \tau \subseteq \sigma, \dim(\sigma) = n\}$$

Indeed, we prove this property using Lemma 2.2.6. This chain contains exactly one critical cell, $(\bar{\tau};\tau)$, and a set of secondary cells. In addition, its boundary does not contain any primary cell:

$$\begin{aligned}\partial'\left(\sum_{\sigma\in I}(\sigma;\tau)\right) &= \sum_{\sigma\in I}\partial'(\sigma;\tau) = \sum_{\sigma\in I}\left(\sum_{\tau'\subseteq\sigma}\langle\partial(\tau'),\tau\rangle\cdot(\sigma;\tau') + \sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot(\sigma';\tau)\right)\\ &= \sum_{\sigma\in I}\sum_{\tau'\subseteq\sigma}\langle\partial(\tau'),\tau\rangle\cdot(\sigma;\tau') + \sum_{\sigma\in I}\sum_{\sigma'\supseteq\tau}\langle\partial(\sigma),\sigma'\rangle\cdot(\sigma';\tau)\\ &= \sum_{\sigma\in I}\sum_{\tau'\subseteq\sigma}\langle\partial(\tau'),\tau\rangle\cdot(\sigma;\tau') + 0\end{aligned}$$

The first term contains only critical or secondary cells, because $\dim(\sigma) = n$. The second term cancels out because each $(\sigma';\tau)$ appears twice, one for each of the two cofaces σ of dimension n of σ' (having 1 or strictly more than 2 cofaces would contradict the fact that S represents a manifold). Therefore,

$$\begin{aligned}d(\bar{\tau};\tau) = \mathtt{j}_C\circ\partial'\circ g(\bar{\tau};\tau) &= \mathtt{j}_C\left(\sum_{\sigma\in I}\sum_{\tau'\subseteq\sigma}\langle\partial(\tau'),\tau\rangle\cdot(\sigma;\tau')\right)\\ &= \mathtt{j}_C\left(\sum_{\tau'}\langle\partial(\tau'),\tau\rangle\cdot g(\bar{\tau}';\tau')\right)\\ &= \sum_{\tau'}\langle\partial(\tau'),\tau\rangle\cdot(\bar{\tau}';\tau') \qquad \text{because } \mathtt{j}_C g = id\end{aligned}$$

Second, on (K^c,∂), the coboundary ∂^* is given by $\partial^*(\tau) = \sum_{\tau'}\langle\partial(\tau'),\tau\rangle\cdot\tau'$. Hence, we have

$$\vee\circ d(\bar{\tau};\tau) \;=\; \sum_{\tau'}\langle\partial(\tau'),\tau\rangle\cdot\tau' \;=\; \partial^*(\tau) \;=\; \partial^*\circ\vee(\bar{\tau};\tau)$$

As a result, $\mathtt{j}_{K^c}\circ\vee\circ\mathtt{i}_C$ is a chain map from $(\operatorname{Span} C, d)$ to $(K^c,\partial)^*$. Moreover, it is an isomorphism, as it associates $(\bar{\tau};\tau)$ to τ. Thus, $\vee$ induces an isomorphism $H\vee$ from $H_{n-q}(\operatorname{Span} C)$ to $H^q(K^c)$.

As (f,g,h) is a reduction from $(\mathsf{sc}(K^c),\partial')$ to $(\operatorname{Span} C, d)$, they have isomorphic groups. All in all, we have the following isomorphisms:

$$H_{n-q}(\mathsf{sc}(K^c)) \xrightarrow{Hf} H_{n-q}(\operatorname{Span} C) \xrightarrow{H\vee} H^q(K^c) \qquad \square$$

We can now prove the Alexander duality in terms of singular homology.

Theorem 4.2.13 (Singular Alexander duality for simplicial complexes)**.** *Let (S,∂) be a*

triangulated n-sphere and $(K,\partial) \subset (S,\partial)$ be a simplicial subcomplex such that $K \neq \{0\}, S$. Then, for every dimension q, $H^q(|S|\backslash|K|) \approx H^q((K^c)^*)$. This implies the geometric version of Alexander duality:

$$\tilde{H}_q(|K|) \approx \tilde{H}^{n-q-1}(|S|\backslash|K|)$$

Proof. We first note that $|S|\backslash|K| = |\mathscr{S}(S)|\backslash|\mathscr{S}(K)|$. Hence we have the following isomorphims, which are summarized in the following diagram:

$$\begin{aligned}
H^q(|S|\backslash|K|) &= H^q(|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|) \\
&\approx H^q(|\mathsf{sc}(K^c)|) && \text{by Lemma 4.2.10} \\
&\approx H^q(\mathsf{sc}(K^c)) && \text{as } \mathsf{sc}(K^c) \text{ is geometric (Lemma 4.2.9)} \\
&\approx H_{n-q}(K^c) && \text{by Proposition 4.2.11} \\
&\approx H^q((K^c)^*) && \text{by duality}
\end{aligned}$$

$$\begin{array}{ccccccc}
H^q(|S|\backslash|K|) & & & & & & H^q((K^c)^*) \\
\downarrow{\scriptstyle id} & & & & & & \uparrow{\scriptstyle id} \\
H^q(|\mathscr{S}(S)|\backslash|\mathscr{S}(K)|) & \xrightarrow[h]{\text{homotopy}} & H^q(|\mathsf{sc}(K^c)|) & \xrightarrow{\text{geometric complex}} & H^q(\mathsf{sc}(K^c)) & \xrightarrow{H(\vee \circ f)} & H_{n-q}(K^c)
\end{array}$$

Hence, taking the reduced homology, we obtain the wanted result:

$$\tilde{H}^{n-q-1}(|S|\backslash|K|) \approx \tilde{H}^{n-q-1}((K^c)^*) \approx \tilde{H}_q(K) \qquad \text{by Theorem 4.2.8}$$ □

4.2.6. Homology quartets

In this subsection, we describe how we can use the proof of Corollary 4.2.7 to represent the holes of a geometric complex.

Homology and cohomology generators are used for displaying the holes in discrete topological spaces such as a surface mesh or a point cloud sampling a manifold, see for instance T. K. Dey, Fan, et al. 2013; Obayashi 2017. Other representations includes the holes balls of Gazull, Bac, et al. 2023 (see Chapter 5) or hole fillings by Wood et al. 2004.

Our method represents the q-hole of a geometric complex K embedded in $\mathbb{R}^n$, with four different elements, each carrying a different intuitive meaning (see Figure 4.14 for an illustration):

1. a q-homology generator of K, which intuitively "surrounds" the hole;
2. a q-cohomology generator of K, which intuitively "cuts" the hole;
3. a $(q+1)$-homology generator of K^c, which intuitively "fills" the hole;
4. a $(q+1)$-cohomology generator of K^c, which intuitively "passes through" the hole.

Where K^c is the complement of K.

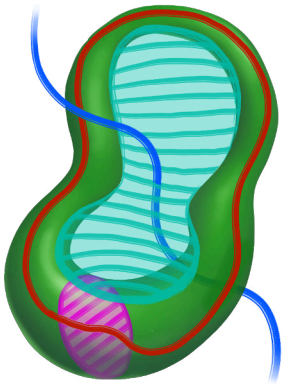


Figure 4.14 – An illustration of the intuition behind homology quartet. The red, pink, blue and cyan elements respectively surround, cut, fill, and pass through the hole.

Formally, if we consider a geometric complex (K,∂) embedded in $\mathbb{R}^n$, let (R,∂) be a triangulation of $\mathbb{R}^n$ such that $(K,\partial) \subset (R,\partial)$.

Recall that the proof of Theorem 4.2.6 considers a pair of subcomplexes (K^0,∂) and (K^1,∂) of (L,∂), L being acyclic and K^1 closed in L under ∂. In our setting, we can choose a 0-cell τ_0 in K, and define $L := R\backslash\tau_0$, $K^1 := K\backslash\tau_0$, $K^0 := L\backslash K^1$.

Now, as for Theorem 4.2.6, we compute two perfect HDVFs $X^0 = (P^0,S^0,C^0)$ and $X^1 = (P^1,S^1,C^1)$ for (K^0,∂) and (K^1,∂), respectively. Next, we choose D to be an invertible map from $\operatorname{Span} C^0_{q+1}$ to $\operatorname{Span} C^1_q$, which is always possible by Theorem 4.2.6.

Then, using D and the isomorphism between homology and cohomology groups, a q-hole of K^1, which is associated to an element $c \in C^1_q$ (see Lemma 1.6.2), can be represented with the four following chains:

1. $g^1_q(c)$, a q-homology generator of $K^1 = K\backslash\tau_0$;
2. ${f^1}^*_q(c)$, a q-cohomology generator of $K^1 = K\backslash\tau_0$;
3. $g^0_{q+1} \circ D^{-1}(c)$, a $(q+1)$-homology generator of $K^0 = K^c$;
4. ${f^0}^*_{q+1} \circ D^{-1}(c)$, a $(q+1)$-cohomology generator of $K^0 = K^c$.

We refer to these four chains as a *homology quartet*.

In the general case, the choice of homology quartet is not unique, and depends on the choice of X^0, X^1 and D. One can choose $D = D^{X^1}_{X^0}$, but other invertible maps are also possible. For instance, every completion process of the HDVF $X = X^0 \sqcup X^1$ provides a possible D. This will be studied in Section 4.4, where we focus on the D that can be obtained from a general HDVF completion.

> *Remark.* It is important to note that there are irrelevant choices for D. Indeed, when D is not obtained from the process described above, it could potentially associates unrelated holes, yielding an incoherent quartet. For a radical example, an incoherent quartet could comprise a generator surrounding a hole, and a generator filling another hole that is not on the same component. Hence, choosing D as a

random invertible matrix is inappropriate. A relevant choice is a *constructible D*, see Proposition 4.4.4.

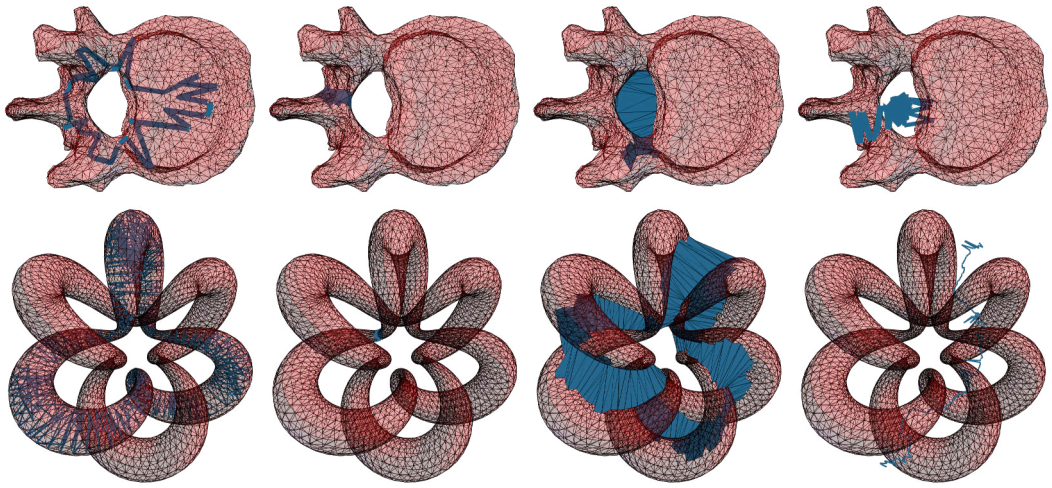


Figure 4.15 – Homology quartets of a 1-hole, for two objects: a bone (above) and a link (below). In order, the columns represent the 1-homology generator, the 1-cohomology generator, the 2-homology generator of the complement, and the 2-cohomology generator of the complement. For clarity and to match with the intuition, the 1-cohomology generator is represented using the cofaces of its cells. In addition, the 2-cohomology generator of the complement is represented using its dual 1-cells.

For practical experiments, we choose the matrix D from a standard completion process of $X = X^0 \sqcup X^1$.

Precisely, performing $\mathtt{A}_\pi^\sigma$ operations on the HDVF X until it is perfect, with $(\sigma, \pi) \in C^0_{q+1} \times C^1_q$, produces a bijection Ψ from C^0_{q+1} to C^1_q. This bijection naturally gives an invertible map $D : \operatorname{Span} C^0_{q+1} \to \operatorname{Span} C^1_q$, which is a permutation matrix (when expressed in the canonical basis given by C^0 and C^1). The choice of D to be a permutation matrix ensures that the cycles and cocycles of the quartet are elementary, see Proposition 2.3.2.

All in all, we proceeded as follows. Given a simplicial complex K embedded in $\mathbb{R}^3$, we embed it in a triangulated 3-ball, which defines an acyclic complex when we subtract a 0-cell. We then computed Ψ as previously explained, and displayed the homology quartets associated with every critical cell γ of K. Precisely, we represent the chains $g^1(\gamma)$, ${f^1}^*(\gamma)$, $g^0 \circ \Psi(\gamma)$ and ${f^0}^* \circ \Psi(\gamma)$.

Some results are presented in Figure 4.15, and can be visualized in 3D online[4]. In addition, implementation of the methods are currently in development and will soon be released in a CGAL package dedicated to HDVFs (The CGAL Project 2025).

[4] https://pageperso.lis-lab.fr/aldo.gonzalez-lorenzo/papers/alexander-duality/quartets.html

4.3. Homology Configurations

4.3.1. Motivations

Section 4.2 illustrates the fact that HDVFs provide explicit constructions of classical homological results, e.g. they produce explicit isomorphisms for Alexander duality, and more generally for combinatorial duality (Theorem 4.2.6).

Another simpler example is the duality between homology and cohomology. Through associated perfect reductions, HDVFs also allow us to construct an explicit isomorphism between the homology group of dimension q of a complex and the corresponding cohomology group. Given (f,g,h) the reduction of a perfect HDVF, these isomorphisms are given by

$$[\cdot]^q \circ f^* \circ f \circ s_q : H_q(K) \to H^q(K) \qquad [\cdot]_q \circ g \circ g^* \circ s^q : H^q(K) \to H_q(K)$$

This naturally raises the following question: can all such isomorphisms be obtained through these constructions, or are there combinatorial obstructions?

In the examples mentioned above, it turns out that not all isomorphisms can be obtained this way. Moreover, these combinatorial obstructions appear to be related to topological properties. This leads to a more general question: which isomorphisms can be constructed using HDVFs? Do they bear any information on the spaces?

Homology configurations were initially introduced in an attempt to address these questions, which is the topic of Section 4.4. However, the properties of homology configurations revealed that they are an object of study in their own right.

Another motivation for introducing homology configurations is that the space of homology computations depends more strongly on the combinatorics of the complex than on its topology, see Section 3.3.4. For instance, the number of possible HDVFs explodes when the cells of a complex are subdivided, although this operation preserves the topology. Thus, adopting a higher point of view, grouping together similar homology computations, becomes relevant.

This section is organized as follows. We begin by introducing the definition of *homology configurations*, together with the intuition behind them. We then establish several structural properties, including the relationship between homology and cohomology configurations, their behavior under HDVF operations, and certain subtleties that arise in maximal dimension. Next, we consider geometric complexes, and investigate the combinatorial invariance of homology configurations in maximal dimension, namely the fact that they are preserved under subdivisions of the complex. We then combine this result with the *Hauptvermutung* to show that homology configurations provide topological invariants for low-dimensional geometric complexes. Finally, we discuss the computation of homology configurations, and present some results.

4.3.2. Definition and intuition

This chapter only presents homology configurations over the field $\mathbb{Z}/2\mathbb{Z}$. Homology configurations can also be defined over other fields. However, in this case, some technicalities arise, and some of our arguments are only valid for $\mathbb{Z}/2\mathbb{Z}$ (see for instance Lemmas 4.3.2 and 4.3.7 and Definition 4.3.2). Homology configurations over $\mathbb{Z}$ constitute a major challenge.

Notations. Given a complex (K,∂), denote $\mathbf{HDVF}^*(K)$ the set of perfect HDVF on (K,∂). For every $q \geq 0$, denote also $\mathbf{TP}_q(K)$ the set of q-dimensional tri-partitions on (K,∂). Finally, denote $\mathbf{Expl}_q(K)$ the set of q explicit homology bases and $\mathbf{Expl}^q(K)$ the set of q explicit cohomology bases of (K,∂).

First, we define an equivalence relation over the explicit bases of a fixed complex (K,∂). Two explicit q-homology (resp. q-cohomology) bases are equivalent if and only if they represent the same basis of the homology group $H_q(K)$ (resp. cohomology group $H^q(K)$)[5].

Formally, given $\{\mathtt{g}_i\}_{i=1}^{\beta_q}$ and $\{\mathtt{g}'_i\}_{i=1}^{\beta_q}$ in $\mathbf{Expl}_q(K)$

$$\{\mathtt{g}_i\}_{i=1}^{\beta_q} \sim_q \{\mathtt{g}'_i\}_{i=1}^{\beta_q} \iff \forall i,\ [\mathtt{g}_i]_q = [\mathtt{g}'_{\phi(i)}]_q \quad \text{with } \phi \text{ a permutation of indices}$$

Similarly, given $\{\mathtt{f}_i\}_{i=1}^{\beta_q}$ and $\{\mathtt{f}'_i\}_{i=1}^{\beta_q}$ in $\mathbf{Expl}^q(K)$

$$\{\mathtt{f}_i\}_{i=1}^{\beta_q} \sim^q \{\mathtt{f}'_i\}_{i=1}^{\beta_q} \iff \forall i,\ [\mathtt{f}_i]^q = [\mathtt{f}'_{\phi(i)}]^q \quad \text{with } \phi \text{ a permutation of indices}$$

Intuitively, two explicit bases are equivalent if they represent homological holes the same way.

In Section 2.3, it was shown that every explicit basis is induced by a perfect HDVF (respectively, by a tri-partition), and conversely that every perfect HDVF (respectively, tri-partition) induces explicit bases. This correspondence allows us to transfer the equivalence relation to perfect HDVFs and tri-partitions. Two perfect HDVFs (respectively, tri-partitions) are said to be equivalent if their induced explicit bases are equivalent:

- Given $X = (P,S,C)$ and $X' = (P',S',C')$ in $\mathbf{HDVF}^*(K)$ (with associated reductions (f,g,h) and (f',g',h'))

$$X \sim_q X' \iff \{g(\gamma)\}_{\gamma\in C_q} \sim_q \{g'(\gamma)\}_{\gamma\in C'_q}$$
$$X \sim^q X' \iff \{f^*(\gamma)\}_{\gamma\in C_q} \sim^q \{f'^*(\gamma)\}_{\gamma\in C'_q}$$

- Given $X = (P,S,C)$ and $X' = (P',S',C')$ in $\mathbf{TP}_q(K)$ (with associated maps z_q, z^q and

[5] In arbitrary fields, one needs to consider that two bases are equivalent if and only if the homology classes of their generators are equal **up to a coefficient**.

z'_q, z'^q)

$$X \sim_q X' \iff \{z_q(\gamma)\}_{\gamma\in C} \sim_q \{z'_q(\gamma)\}_{\gamma\in C'}$$
$$X \sim^q X' \iff \{z^q(\gamma)\}_{\gamma\in C} \sim^q \{z'^q(\gamma)\}_{\gamma\in C'}$$

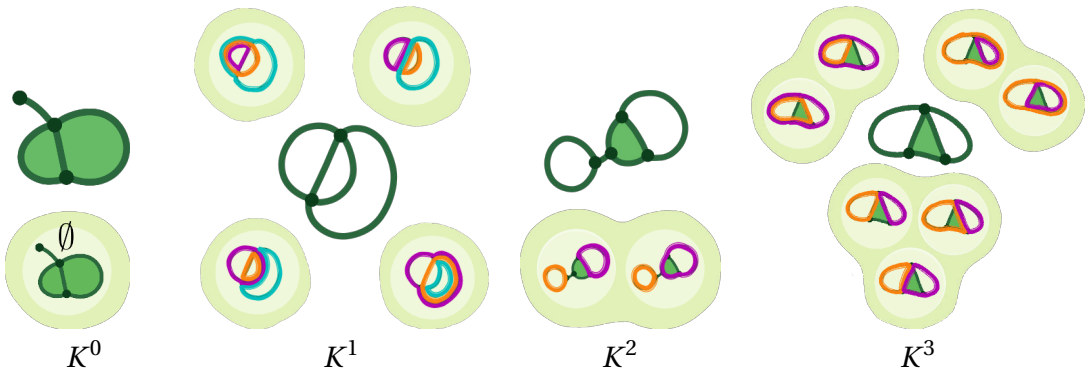


Figure 4.16 – Four complexes and their associated 1-homology configurations. Each homology configuration is represented by its corresponding equivalent explicit 1-homology bases. For instance, $\mathbf{Conf}_1(K^3)$ is composed of three configurations: the bottom configuration is composed of three equivalent explicit bases, in which the two generators respectively surround the left (in orange) and the right holes (in purple). In the top-left configuration, explicit bases are composed of a generator surrounding the left hole (in orange) and the other surrounding both holes (in purple). In the top-right configuration, explicit bases are composed of a generator surrounding the right hole (in purple) and the other surrounding both holes (in orange).

We can then define the *homology configurations*, which correspond to the equivalence classes of $\sim_q$ and $\sim^q$.

Definition 4.3.1 ((co)homology configurations)**.** Given a complex K and a dimension q, we define $\mathbf{Conf}_q(K)$, the *q-homology configurations* of K, and $\mathbf{Conf}^q(K)$, the *q-cohomology configurations* of K, as

$$\mathbf{Conf}_q(K) := \mathbf{HDVF}^*(K)/\sim_q \qquad \mathbf{Conf}^q(K) := \mathbf{HDVF}^*(K)/\sim^q$$

Because of the results of Section 2.3, homology configurations can alternatively be defined in terms of explicit bases or tri-partitions:

$$\mathbf{Conf}_q(K) \approx \mathbf{Expl}_q(K)/\sim_q \approx \mathbf{TP}_q/\sim_q \qquad \mathbf{Conf}^q(K) \approx \mathbf{Expl}^q(K)/\sim^q \approx \mathbf{TP}_q/\sim^q$$

See Figures 4.16 and 4.17 for examples of homology configurations.

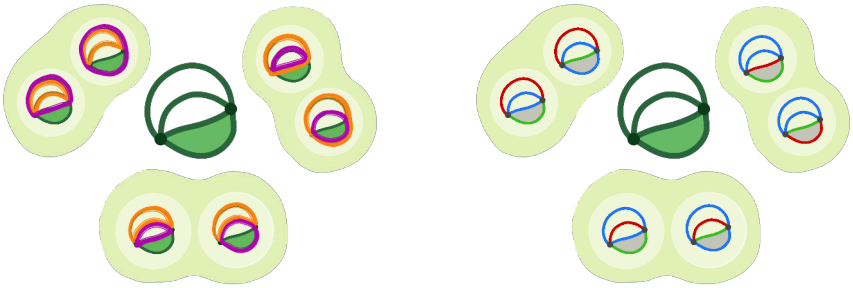


Figure 4.17 – A complex and its associated 1-homology configurations. On the left, each homology configuration is represented by its equivalent explicit 1-homology bases. On the right, each homology configuration is represented by its equivalent 1-tri-partitions.

Remark. In general, the set $\mathbf{Conf}_q(K)$ doesn't have a natural algebraic structure, so writing "$\mathbf{Conf}_q(K) \approx X$" simply means that there exists a bijection between $\mathbf{Conf}_q(K)$ and X. Later, in Section 4.3.6, we will equip $\mathbf{Conf}_{\dim K}(K)$ with a graph structure.

Another way of understanding homology configurations is to consider $\mathbf{Conf}_q(K)$ as the set of bases of $H_q(K)$ that can be represented as explicit bases.

On the one hand, the definitions of homology configurations in terms of HDVFs and tri-partitions allow us to derive interesting combinatorial properties, as discussed in Section 4.3.3. On the other hand, one can consider that the definition in terms of explicit bases is more natural and more precise, in the sense that there are fewer explicit bases than HDVFs and tri-partitions. This observation is highlighted by the following lemma, which complement the illustration in Figure 3.4.

Lemma 4.3.1. *We have the following sequence of surjections:*

$$\mathbf{HDVF}^*(K) \twoheadrightarrow \mathbf{TP}_q(K) \twoheadrightarrow \mathbf{Expl}_q(K) \twoheadrightarrow \mathbf{Conf}_q(K)$$

Proof. Using Proposition 2.2.5, there exists a surjection from $\mathbf{HDVF}^*(K)$ to $\mathbf{TP}_q(K)$. Moreover, using Theorem 2.3.5 with tri-partitions, there is also a surjection from $\mathbf{TP}_q(K)$ to $\mathbf{Expl}_q(K)$. By quotient, there exists a surjection from $\mathbf{Expl}_q(K)$ to $\mathbf{Conf}_q(K) = \mathbf{Expl}_q(K)/\sim_q$. □

4.3.3. Properties

Relation between homology and cohomology configurations

The following lemma is useful as it links the homology configurations to algebraic and combinatorial properties on the reductions associated to HDVF.

Lemma 4.3.2. *Given $X = (P, S, C)$ and $X' = (P', S', C')$ two perfect HDVF over K (with*

associated reduction (f,g,h) *and* (g',f',h')*) we have:*

$$X \sim_q X' \iff f'_q \circ g_q : \operatorname{Span} C_q \to \operatorname{Span} C'_q \text{ is a permutation matrix}$$
$$X \sim^q X' \iff g'^*_q \circ f^*_q : \operatorname{Span} C_q \to \operatorname{Span} C'_q \text{ is a permutation matrix}$$

Proof. We will omit the q subscripts in the proof for clarity.

- $\Longrightarrow$. Firstly, $f'g$ is invertible:

  $$\begin{aligned} f'g \circ fg' &= f'(id - \partial h - h\partial)g' \\ &= f'g' - f'\partial hg' - f'h\partial g' \\ &= id - 0 - 0 \quad \text{because } f'g' = id,\ f'\partial = 0,\ \partial g = 0 \text{ as } X \text{ and } X' \text{ are perfect} \end{aligned}$$

  Similarly, $fg' \circ f'g = id$.

  It remains to prove that each column has no more than one non-zero value: given $\gamma \in C$, there exists $c' \in \operatorname{Span} C'$ such that $c' := f'g(\gamma)$. Let us prove that c' is composed of only one cell.

  On one hand, by applying g' to c' we get $g'(c') = g(\gamma) - \partial h'g(\gamma) - h'\partial g(\gamma) = g(\gamma) - \partial(h'g(\gamma))$. Hence, $[g(\gamma)]_q = [g'(c')]_q$. On the other hand, by definition of $\sim_q$, there exists $\gamma' \in C'$ such that $[g(\gamma)]_q = [g'(\gamma')]_q$. As a result, $[g'(\gamma')]_q = [g'(c')]_q$. As $([g'(\tau)]_q)_{\tau \in C'}$ is a basis of $H_q(K)$, we have that $\gamma' = c'$.

- $\Longleftarrow$. Let $\gamma \in C$. We show that $[g(\gamma)]_q \in [\cdot]_q \circ g'_q(C'_q)$.

  $f'g$ is a permutation matrix from $\operatorname{Span} C$ to $\operatorname{Span} C'$ so there exists a cell $\gamma' \in C'$ such that $\gamma' = f'g(\gamma)$. By applying g' we get $g'(\gamma') = g(\gamma) - \partial h'g(\gamma) - h'\partial g(\gamma) = g(\gamma) - \partial(h'g(\gamma))$, so we have $[g(\gamma)]_q = [g'(\gamma')]_q$. Hence, $[g(\gamma)]_q \in [\cdot]_q \circ g'_q(C'_q)$ and $[\cdot]_q \circ g_q(C_q) \subseteq [\cdot]_q \circ g'_q(C'_q)$.

  The other inclusion is obtained because fg' is also a permutation matrix (as it is equal to $(f'g)^{-1}$).

The proof for cohomology is the dual of the one for homology. □

Intuitively, the map $\phi_q = f'_q \circ g_q$ associates to a critical cell of a HDVF its corresponding critical cells in another HDVF that represents that same hole. Lemma 4.3.2 states that when two HDVFs are in the same configuration, ϕ_q is a pairing between critical cells (equivalently it is a permutation matrix, when expressed in the basis given by the critical cells). This result implies the following corollary, which relates the (co)homology maps (i.e. the isomorphisms of Lemma 1.6.2) of two distinct perfect HDVFs.

Corollary 4.3.3. *Let* X *and* X' *be two perfect HDVFs. Denote* $\phi_q = f'_q \circ g_q : \operatorname{Span} C \to \operatorname{Span} C'$*, and* $\phi^q = g'^*_q \circ f^*_q : \operatorname{Span} C \to \operatorname{Span} C'$*. The following properties hold:*

1. *if* $X \sim_q X'$*, then* $\phi_q = \phi^q$*.*
2. $[\cdot]_q g_q = [\cdot]_q g'_q \circ \phi_q$*.*

3. $[\cdot]^q f_q^* = [\cdot]^q f_q'^* \circ \phi^q$.

Proof.

1. Firstly, $(f_q' g_q)^{-1} = f_q g_q'$. Indeed, $f'gfg' = f'g' - f'\partial hg' - f'h\partial g' = id - 0 - 0$ by reduction properties (see Definition 1.4.2). $f_q' g_q$ is a permutation matrix by Lemma 4.3.2, and the inverse of a permutation matrix equals its transposed. Hence, we have:
$$f_q' g_q = (f_q g_q')^{-1} = (f_q g_q')^* = g_q'^* \circ f_q^*$$
2. By reduction properties we have $g'f'g = g - \partial h'g - h'\partial g = g - \partial h'g$. Hence, as $[\cdot]\partial = 0$, we obtain $[\cdot]_q g_q = [\cdot]_q g_q' \circ f_q' g_q$.
3. Same proof as *2.* by duality. □

This directly implies the following proposition, which gathers homology and cohomology configurations.

Proposition 4.3.4. *Given two perfect HDVFs X and X' we have $X \sim_q X' \iff X \sim^q X'$. In particular we have* $\mathbf{Conf}_q(K) = \mathbf{Conf}^q(K)$.

Proof. Using Corollary 4.3.3(1.), Lemma 4.3.2, and the fact that the transpose of a permutation matrix is a permutation matrix, we obtain:

$$\begin{aligned} X \sim_q X' &\iff f_q' \circ g_q : \operatorname{Span} C_q \to \operatorname{Span} C_q' \text{ is a permutation matrix} \\ &\iff g_q'^* \circ f_q^* : \operatorname{Span} C_q' \to \operatorname{Span} C_q \text{ is a permutation matrix} \\ &\iff X \sim^q X' \end{aligned}$$
□

In other words, $\sim_q = \sim^q$ so homology and cohomology configurations are the same. Proposition 4.3.4 can be understood as a complement of Section 2.3.4, with a higher point of view. Indeed, Propositions 2.3.9 and 2.3.10 established a combinatorial characterization of compatible explicit bases, whereas Proposition 4.3.4 implies that compatible explicit bases must be in the same configuration.

As a result, whatever the construction we use (HDVF, tri-partitions, explicit bases, with homology or cohomology) we get the same set of configurations, up to bijection. Hence, to shorten notation, we simply refer to $\mathbf{Conf}_q(K)$ as the *q-configurations*, omitting to mention homology or cohomology.

Configurations and Operations

Here, we investigate how HDVF operations act on homology configurations. The following proposition provides a sufficient condition for an operation to preserve the configuration. Proposition 4.3.6 is the reciprocal of Proposition 4.3.5, but only for `M` and `W` operations, as the case of $\Diamond$ remains open.

Proposition 4.3.5. *Let (z_q, z^q) be the canonical maps of a perfect HDVF X.*

1. *If* $z_q(\pi-\gamma)\in\operatorname{im}\partial$ *(equivalently,* $f(\pi-\gamma)=0$*), then* $X\sim_q \mathtt{M}_\pi^\gamma X$.
2. *If* $z^q(\sigma-\gamma)\in\operatorname{im}\partial^*$ *(equivalently,* $g^*(\sigma-\gamma)=0$*), then* $X\sim_q \mathtt{W}_\sigma^\gamma X$.
3. *If* $z_q(\pi)\in\operatorname{im}\partial$ *or* $z^q(\sigma)\in\operatorname{im}\partial^*$ *(equivalently,* $f(\pi)=0$ *or* $g^*(\sigma)=0$*), then* $X\sim_q \Diamond_\pi^\sigma X$.

Proof. Let $X=(P,S,C)$ be a perfect with reduction (f,g,h). In all the following cases, we denote by $X'=(P',S',C')$ the perfect HDVF obtained after performing an operation ($\mathtt{M}$, $\mathtt{W}$ or $\Diamond$) on X. Let (f',g',h') be its associated reduction and (z'_q,z'^q) its canonical maps.

1. $\mathtt{M}_\pi^\gamma$. Suppose $z_q(\pi-\gamma)\in\operatorname{im}\partial$ First, as $\pi\in C'_q$ and $\gamma\in C_q$, consider $g(\gamma)$ and $g'(\pi)$. We have:

$$\begin{aligned} g'(\pi)-g(\gamma) &= z'_q(\pi)-z_q(\gamma)\\ &= z_q(\pi)-z_q(\gamma)\in\operatorname{im}\partial \qquad \text{by Lemma 1.6.8}\end{aligned}$$

   Hence, $[g(\gamma)]=[g'(\pi)]$.

   Second, let $\tau\in C'_q\backslash\{\sigma\}\subseteq C_q$. We have:

$$\begin{aligned} g(\tau)-g'(\tau) &= z_q(\tau)-z'_q(\tau)\\ &= z_q(\tau)-z_q(\tau)=0\in\operatorname{im}\partial \qquad \text{by Lemma 1.6.8}\end{aligned}$$

   Hence, $[g(\tau)]=[g'(\tau)]$. All in all, $\{g(\tau)\}_{\tau\in C}=\{g'(\tau)\}_{\tau\in C'}$ so $X\sim_q X'$.
2. $\mathtt{W}_\sigma^\gamma$. This case is the dual of the previous case. We obtain $X\sim^q X'$, which is equivalent to $X\sim_q X'$ by Proposition 4.3.4.
3. $\Diamond_\sigma^\gamma$. In this case, both the previous proofs work. □

Conversely, the following proposition provide a necessary condition on an operation so that it preserves the configuration.

Proposition 4.3.6. *Let* (z_q,z^q) *be the canonical maps of a perfect HDVF* X.

1. *If* $X\sim_q \mathtt{M}_\pi^\gamma X$*, then* $z_q(\pi-\gamma)\in\operatorname{im}\partial$ *(equivalently,* $f(\pi-\gamma)=0$*).*
2. *If* $X\sim_q \mathtt{W}_\sigma^\gamma X$*, then* $z^q(\sigma-\gamma)\in\operatorname{im}\partial^*$ *(equivalently,* $g^*(\sigma-\gamma)=0$*).*

Proof.

1. $X\sim_q \mathtt{M}_\pi^\gamma X$. Consider $X'=(P',S',C'):=\mathtt{M}_\pi^\gamma X\sim_q X=(P,S,C)$. Let z_q and z'_q be their associated canonical cycle maps.

   As $X'\sim_q X$, we have for all $\tau\in C$, $z'_q(\phi(\tau))-z_q(\tau)\in\operatorname{im}\partial$, with ϕ a bijection from C to C'. Lemma 1.6.8 implies that $z'_q=z_q$. Hence, let $\tau\in C\backslash\{\gamma\}=C'\backslash\{\pi\}$, we have $z'_q(\tau)=z_q(\tau)$. As a result, $\phi(\tau)=\tau$, otherwise $\left\{[z'_q(\tau)]\right\}_{\tau\in C'}$ would not be linearly independent.

As ϕ is a pairing from C to C', the two remaining unpaired cells are $\gamma \in C$ and $\pi \in C'$, so $\phi(\gamma) = \pi$. Therefore, we obtain:

$$\begin{aligned} z_q(\pi - \gamma) &= z_q(\pi) - z_q(\gamma) = z'_q(\pi) - z_q(\gamma) && \text{as } z_q = z'_q \\ &= z'_q(\phi(\gamma)) - z_q(\gamma) \in \operatorname{im}\partial && \text{as } \phi(\gamma) = \pi \end{aligned}$$

2. $X \sim_q \mathtt{W}^{\gamma}_{\sigma} X$. The proof for `W` is similar to the one for `M` by duality. □

All in all, we obtain a simple necessary and sufficient condition for operations `M` and `W` so that they preserve the configurations. Later, in Section 4.3.6, we will also show that `W` operations can be extended to $\mathbf{Conf}_n(K)$ itself, with n being the maximal dimension of K.

Subtleties in maximal dimension

We conclude this introduction to homology configurations with two lemmas highlighting the particular behavior that arise when considering the configurations in the maximal dimension of the complex. These lemmas will be used in the next subsections, about combinatorial/topological invariance and computations of homology configurations in maximal dimension.

Remark. By duality, the following lemmas also hold for the minimal dimension (with the corresponding dual concepts). Though, the minimal dimension is often uninteresting in practice, so we only state these results for the maximal dimension, as it is more compelling.

The first lemma states that there exists exactly one explicit basis per homology configuration in maximal dimension.

Lemma 4.3.7. *If K is a n-complex, then we have* $\mathbf{Expl}_n(K) \approx \mathbf{Conf}_n(K)$.

Proof. Let $\mathscr{B} = \{\mathrm{g}_i\}_{i=1}^{\beta_n(K)}$ and $\mathscr{B}' = \{\mathrm{g}'_i\}_{i=1}^{\beta_n(K)}$ in $\mathbf{Expl}_n(K)$ such that $\mathscr{B} \sim_n \mathscr{B}'$.

Fix i, there exists j such that $[\mathrm{g}_i]_n = [\mathrm{g}'_j]_n$. Hence, there exists a $n+1$ chain y such that $\mathrm{g}_i = \mathrm{g}'_j + \partial y$. However, there is no $n+1$ cell in K so $y = 0$, which implies $\mathrm{g}_i = \mathrm{g}'_j$. As a consequence, $\mathrm{g}_i \in \mathscr{B}'$. We can obtain a similar result for every other g_i and g'_i, which implies $\mathscr{B}' = \mathscr{B}$.

As a result, $\sim_n$ coincide with $=$, thus, $\mathbf{Conf}_n(K) = \mathbf{Expl}_n(K)/\sim_n \approx \mathbf{Expl}_n(K)$. □

The second lemma provides a simple criterion for determining whether two HDVFs or tri-partitions belong to the same configuration. This result relies on specific properties of HDVFs in the maximal dimension n of the complex. For instance, if we denote $X = (P, S, C)$ a perfect HDVF and (f, g, h) its associated reduction, then $P_n = \emptyset$. Moreover, the map f_n simply becomes $\mathtt{j}_{C_n}$.

Lemma 4.3.8. *Let $X = (P,S,C)$ and $X' = (P',S',C')$ be two perfect HDVF for a complex (K,∂) of maximal dimension n, and let (f,g,h) and (f',g',h') be their respective reductions. Then*

$$X \sim_n X' \iff \forall \gamma \in C_n,\ |\overline{g_n(\gamma)} \cap C'_n| = 1$$

In particular, if $X = (P,S,C)$ and $X' = (P',S',C')$ are two n-tri-partitions (with canonical cycle map z and z'), $X \sim_n X' \iff \forall \gamma \in C,\ |\overline{z(\gamma)} \cap C'| = 1$

Proof. For the dimension n we have $P_n = P'_n = \emptyset$, hence, $f_n = \mathtt{j}_{C_n}$ and $f'_n = \mathtt{j}_{C'_n}$.

To begin with, we prove that $\mathtt{j}_{C'_n} \circ g_n$ is a permutation matrix if and only if $\forall \gamma \in C_n,\ |\overline{g_n(\gamma)} \cap C'_n| = 1$:

- First, if $\mathtt{j}_{C'_n} \circ g_n$ is a permutation matrix, then it has exactly one non-zero coefficient in the column associated to a cell $\gamma \in C_n$, which is equivalent to $\forall \gamma \in C_n,\ |\overline{g_n(\gamma)} \cap C'_n| = 1$.
- Conversely, suppose $\mathtt{j}_{C'_n} \circ g_n$ has exactly one non-zero coefficient in each column. In order to prove that it is a permutation matrix, it remains to prove that it is surjective: let $c' \in \operatorname{Span} C'_n$. We have $g'_n(c') \in \ker \partial \cap K_n \subseteq \operatorname{Span}((g_n(\gamma))_{\gamma \in C_n})$ because $(g_n(\gamma))_{\gamma \in C_n}$ is explicit (by Proposition 2.3.1 2.). Hence, $\exists c \in \operatorname{Span} C_n$ such that $g'_n(c') = g_n(c)$, which implies $c' = \mathtt{j}_{C'_n} g'_n(c') = \mathtt{j}_{C'_n} g_n(c)$.

All in all we have:

$$\begin{aligned} X \sim_n X' &\iff f'_n \circ g_n \quad \text{is a permutation matrix} && \text{by Lemma 4.3.2} \\ &\iff \mathtt{j}_{C'_n} \circ g_n \quad \text{is a permutation matrix} \\ &\iff \forall \gamma \in C_n,\ |\overline{g_n(\gamma)} \cap C'_n| = 1 && \square \end{aligned}$$

Later, we present other properties of configurations in maximal dimension. For instance, Lemmas 4.3.23 and 4.3.24 shows that `W` operations can be extended to these configurations. In the next subsection, we prove that they are preserved under subdivision of the complex.

4.3.4. Combinatorial invariance

This subsection only considers geometric chain complexes. The goal of this subsection is to prove that the number of homology configurations in the maximal dimension is preserved when we subdivide the complex. This result is stated in Theorem 4.3.10. It will later be used to prove Theorem 4.3.21, which states that the number of homology configurations in the maximal dimension is a topological invariant for geometric complexes of dimension less or equal to 3. Note that this is not true in general for dimensions below the maximal one, as illustrated by the simple counterexample in Figure 4.18.

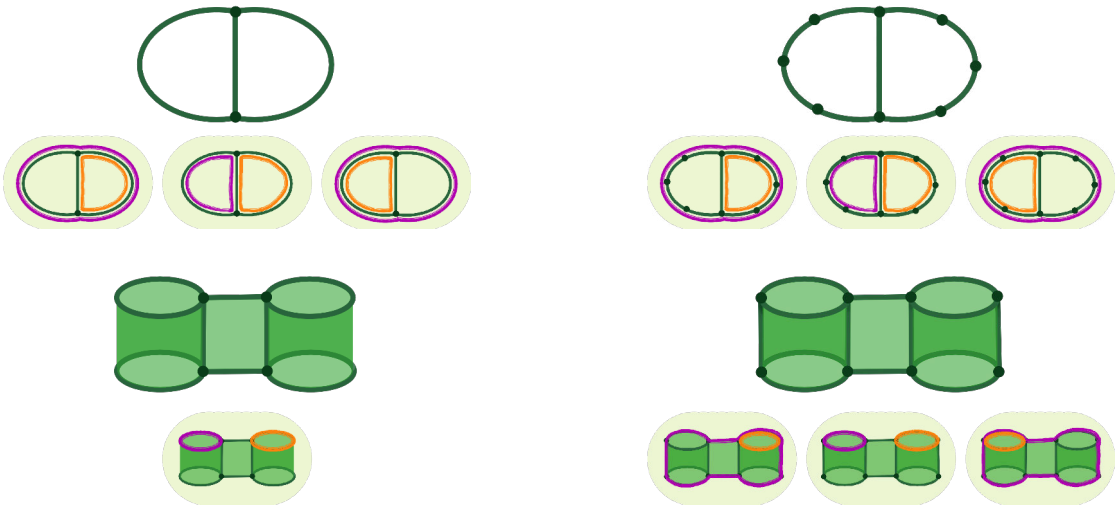


Figure 4.18 – Four complexes and their 1-configurations. Each configuration is represented by an explicit basis. The complexes on the right are subdivisions of the complexes on the left. In the first row, the complexes are of dimension 1 and the 1-configurations are preserved under subdivision, which illustrates Theorem 4.3.10. On the contrary, the complexes in the second row are of dimension 2, and some 1-configurations are created after subdivision. This is a counterexample to the combinatorial invariance of q-configurations, with q lesser than the dimension of the complex.

4.3.4.1. General and elementary subdivisions

First, we define a notion of general subdivision of a complex. We then introduce elementary subdivisions, which intuitively correspond to subdivisions involving a single cell. Subdivisions are essentially combinatorial, but in the present setting, we need to combine them to topology and geometry. Thus, we introduce a definition of subdivisions compatible with both the combinatorial chain complex structure and the geometric structure.

Definition 4.3.2. A *subdivision* of (K,∂) is a geometric complex (K',∂') such that there is a cell inclusion function $i : \overline{K'} \to \overline{K}$ satisfying:

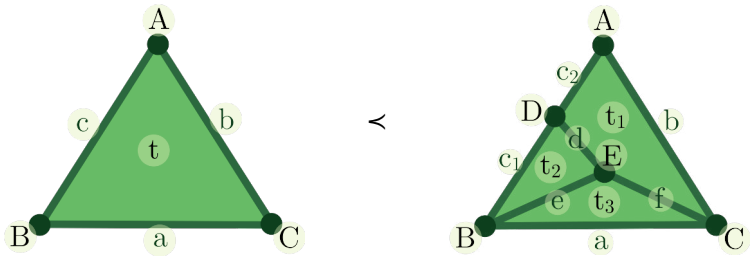
1. $\forall \tau \in \overline{K}$, $|i^{-1}(\tau)| = |\tau|$, which implies that $|K'| = |K|$ so K' and K are complexes representing the same topological space.
2. The following injective map χ (defined on the cells and extended linearly) is a chain complex morphism, i.e. $\partial'\chi = \chi\partial$:

$$\begin{aligned} \chi : K &\longrightarrow K' \\ \tau &\longmapsto \sum_{\substack{\sigma \in i^{-1}(\tau) \\ \dim\sigma = \dim\tau}} \sigma \end{aligned}$$

We write $K \prec K'$ when K' is a subdivision of K, see Figure 4.19 for an example

of subdivision. Note that this definition is not standard, but is general enough to encompass every common subdivision scheme, see for instance Figure 4.20.

Remark. For a proper definition of subdivision with homology over $\mathscr{R} \neq \mathbb{Z}/2\mathbb{Z}$, the formula for χ needs to incorporate coefficients: $\chi(\tau) = \sum_\sigma \varepsilon_\sigma^\tau \sigma$. The coefficients $\varepsilon_\sigma^\tau \in \mathscr{R}$ depend on the boundary ∂' of the subdivision, which in turn depends on the geometric construction (simplicial, CW, cubical or polyhedral complexes) and orientation.



$i^{-1}(\mathrm{A}) = \{\mathrm{A}\}$ $\quad i^{-1}(\mathrm{a}) = \{\mathrm{a}\}$
$i^{-1}(\mathrm{B}) = \{\mathrm{B}\}$ $\quad i^{-1}(\mathrm{b}) = \{\mathrm{b}\}$
$i^{-1}(\mathrm{C}) = \{\mathrm{C}\}$ $\quad i^{-1}(\mathrm{c}) = \{\mathrm{D}, \mathrm{c}_1, \mathrm{c}_2\}$
$i^{-1}(\mathrm{t}) = \{\mathrm{E}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3\}$

$\chi(\mathrm{A}) = \mathrm{A}$ $\quad \chi(\mathrm{a}) = \mathrm{a}$
$\chi(\mathrm{B}) = \mathrm{B}$ $\quad \chi(\mathrm{b}) = \mathrm{b}$
$\chi(\mathrm{C}) = \mathrm{C}$ $\quad \chi(\mathrm{c}) = \mathrm{c}_1 + \mathrm{c}_2$
$\chi(\mathrm{t}) = \mathrm{t}_1 + \mathrm{t}_2 + \mathrm{t}_3$

Figure 4.19 – Example of a general subdivision.

We introduce a special type of subdivision, called *elementary subdivision*, which consists in the subdivision of a single cell into arbitrary many cells.

Definition 4.3.3. An *elementary subdivision* of (K, ∂) is a subdivision (K', ∂') where there exists a $\tau \in \overline{K}$ and $T \subset \overline{K'}$, such that $\overline{K} \setminus \{\tau\} = \overline{K'} \setminus T$, ∂ and ∂' coincide on mutual cells, and

$$i(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in \overline{K'} \setminus T \\ \tau & \text{if } \sigma \in T \end{cases}$$

Using Definition 4.3.2, we have the following properties:

- $|\tau| = |T|$.
- $i^{-1}(\sigma) = \{\sigma\}$ for $\sigma \in \overline{K} \setminus \{\tau\}$, and $i^{-1}(\tau) = T$.
- Hence, $\chi(\sigma) = \sigma$ for $\sigma \in \overline{K} \setminus \{\tau\}$, and $\chi(\tau) = \sum_{\sigma \in T_q} \sigma$ with T_q the q-cells of T, where q is the dimension of τ.
- $\forall x \in K$, $\chi(x) = x - \langle x, \tau \rangle (\tau - \chi(\tau))$. In particular, if $\tau \notin \bar{x}$, $\chi(x) = x$, and $\partial' \chi(\tau) = \partial(\tau)$.
- Denote $K^- := \mathrm{Span}\,(\overline{K'} \setminus T)$. Then, $K = K^- \oplus \mathrm{Span}\,\{\tau\}$ and $K' = K^- \oplus \mathrm{Span}\, T$.

We have the following proposition, which eases our proof as it states that elementary subdivisions generate every subdivision.

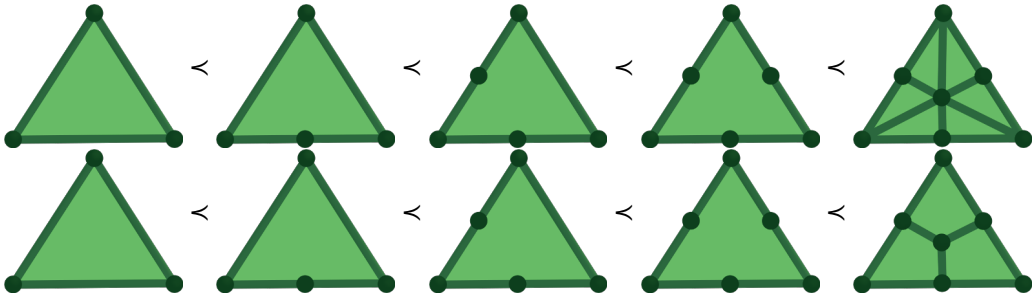


Figure 4.20 – Decomposition of two commonly used subdivision schemes into elementary subdivisions. The first row represents a barycentric division whereas the second row represents a Catmull-Clark subdivision. This later subdivision is the one used in Section 4.2.5 (see Definition 4.2.1). In this case, the inclusion function i is given by $\wedge$.

Proposition 4.3.9. *Every subdivision (K',∂') of a complex (K,∂) can be decomposed as a sequence of elementary subdivisions: $K = K^0 \prec K^1 \prec \ldots \prec K^N = K'$.*

Proof. The idea of the proof is simple: we order the cells of K by increasing dimension, and perform elementary subdivisions cell by cell to transform K into K'. However, the proof is tedious to write properly. We decided to put this proof in appendix to ease the reading of this (already technical) subsection, see Proof of Proposition 4.3.9. □

4.3.4.2. Plan of the proof of combinatorial invariance

We can now prove that homology configurations in maximal dimension are invariant with respect to general subdivision. The remaining of this subsection is dedicated to the proof of the following theorem.

Theorem 4.3.10 (combinatorial invariance of homology configurations)**.** *Let (K,∂) and (K',∂') be two geometric n-complexes such that $K \prec K'$. Then,* $\mathbf{Conf}_n(K) \approx \mathbf{Conf}_n(K')$.

First, Proposition 4.3.9 implies that we can simply consider elementary subdivisions. Moreover, the following lemma proves the combinatorial invariance with respect to subdivisions of lower dimensional cells.

Lemma 4.3.11. *Let p be a dimension and (K',∂') a subdivision of (K,∂) that subdivide only q-cells, with $q \leq p$. Then for all $q > p$,* $\mathbf{Expl}_q(K) = \mathbf{Expl}_q(K')$ *and* $\mathbf{Conf}_q(K) = \mathbf{Conf}_q(K')$.

In other words, a subdivision of cells of dimension lesser or equal to p does not change the explicit bases and the homology configurations of higher dimensions.

Proof. Let $\chi : K \to K'$ be the complex morphism defined following Definition 4.3.2. Denote χ_q the map χ restrained to q-cells. For all q we have $\chi_q \partial_{q+1} = \partial'_{q+1} \chi_{q+1}$.

When $q > p$ we have $\chi_q = id_{K_q}$, so this implies $K_q = K'_q$ and $\partial_{q+1} = \partial'_{q+1}$, so $\operatorname{im}\partial'_{q+1} = \operatorname{im}\partial_{q+1}$. In addition, we obtain $\chi_{q-1}\partial_q = \partial'_q$ and χ is injective so, $\ker\partial'_q = \ker\partial_q$. The definition of q-explicit bases only depends on q-cells, q-cycles and q-boundaries (see Proposition 2.3.1), hence $\mathbf{Expl}_q(K) = \mathbf{Expl}_q(K')$. □

As a result, in order to prove Theorem 4.3.10, it remains to prove that $\mathbf{Conf}_n(K)$ is preserved under elementary subdivision of a cell of maximal dimension. This case is harder, and is presented in both following technical parts.

The setting is the following: let n be the maximal dimension of a complex (K,∂). We consider an elementary subdivision: $(K,\partial) = (K^- \oplus \operatorname{Span}\{\tau\},\partial) \prec (K',\partial') = (K^- \oplus \operatorname{Span} T,\partial)$, with τ an n-cell in K, and T a set of cells of dimension inferior or equal to n. We also define i and χ as in Definition 4.3.3, and denote $t := \chi(\tau) = \sum_{\sigma\in T_n}\sigma$.

The outline of the proof is the following:

1. We build an injection from $\mathbf{Conf}_n(K)$ to $\mathbf{Conf}_n(K')$ (Section 4.3.4.3):
   - We introduce $\tilde{\chi} : \mathbf{Expl}_n(K) \to \mathbf{Expl}_n(K')$, which maps explicit bases to their image under χ.
   - We prove that $\tilde{\chi}$ is well-defined (Lemma 4.3.13).
   - Finally, we show that $\tilde{\chi}$ induces an injection $[\tilde{\chi}]$ from $\mathbf{Conf}_n(K)$ to $\mathbf{Conf}_n(K')$ (Proposition 4.3.12).
2. We build an injection from $\mathbf{Conf}_n(K')$ to $\mathbf{Conf}_n(K)$ (Section 4.3.4.4):
   - We introduce $p : \mathbf{TP}_n(K') \to \mathbf{TP}_n(K)$, which maps tri-partitions of K' to tri-partitions of K.
   - We introduce technical lemmas about how critical cells behave after subdivision (Corollary 4.3.17, which is a consequence of Lemmas 4.3.15 and 4.3.16).
   - We use these lemmas to prove that p is well-defined (Proposition 4.3.18).
   - Then, we show that p preserves the canonical cycles of tri-partitions (Lemma 4.3.19).
   - Finally, this implies that p preserves the configurations, so it induces an injection $[p]$ from $\mathbf{Conf}_n(K')$ to $\mathbf{Conf}_n(K)$ (Proposition 4.3.14).

In the next subsection (Section 4.3.5), we will combine these result with the *Hauptvermutung*, to prove the topological invariance of n-configurations for complexes of dimension 3 or lower.

4.3.4.3. Building an injection from $\mathbf{Conf}_n(K)$ to $\mathbf{Conf}_n(K')$

We define the following function:

$$\begin{aligned}\tilde{\chi} : \mathbf{Expl}_n(K) &\to \mathbf{Expl}_n(K')\\ \{g_i\}_{i=1}^{\beta_n} &\mapsto \{\chi(g_i)\}_{i=1}^{\beta_n}\end{aligned}$$

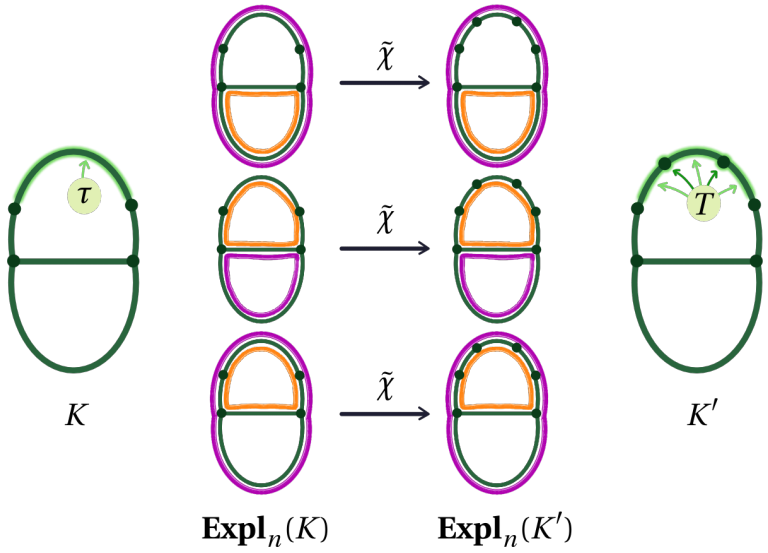


Figure 4.21 – Illustration of the function $\tilde{\chi}$.

Precisely, if τ is a cell of g_i, then it is replaced by t in $\chi(g_i)$, see Figure 4.21 for an illustration. We want to prove the following proposition.

Proposition 4.3.12. *Let $[\tilde{\chi}]$ be the function that associates to every n-configuration of K its corresponding n-configuration of K', following $\tilde{\chi}$. $[\tilde{\chi}]$ is well-defined, and is an injection from* $\mathbf{Conf}_n(K)$ *to* $\mathbf{Conf}_n(K')$

The idea is that $\tilde{\chi}$ transforms an explicit basis of K in to an explicit basis of K' while preserving the homology configuration. We first show that χ is well-defined. The proof is technical and involves the geometric and combinatorial properties of elementary subdivisions. Then, we show χ is injective and conclude the proof of Proposition 4.3.12.

Lemma 4.3.13. *$\tilde{\chi}$ is well-defined i.e. if $\{g_i\}_{i=1}^{\beta_n}$ is an explicit basis of K, then $\{\chi(g_i)\}_{i=1}^{\beta_n}$ is an explicit basis of K'.*

Proof. First, $\chi(g_i)$ is a cycle, because g_i is a cycle and χ a chain morphism. Therefore, $\{\chi(g_i)\}_{i=1}^{\beta_n}$ is a family of β_n cycles of K'. We now show that it is a homology basis i.e. that $\left\{[\chi(g_i)]^{K'}\right\}_{i=1}^{\beta_n}$ is a basis of $H_n(K')$.

As $H_n(K') \approx H_n(K)$ has dimension β_n, it is sufficient to show that $\left\{[\chi(g_i)]^{K'}\right\}_{i=1}^{\beta_n}$ is linearly independent.

[6] Notice the slight difference between K^{χ}, where we consider all the faces, and the down-induced complexes of Definitions 2.3.1 and 2.3.2, where we only consider the $(n-1)$-faces. This difference does not alter the proof, as both complexes have the same n and $(n-1)$-chain groups, and so have isomorphic n-homology groups.

Thus, let $(\varepsilon_i)_{i=1}^{\beta_n}$ be coefficient such that $\sum_{i=1}^{\beta_n} \varepsilon_i [\chi(\mathrm{g}_i)]^{K'} = 0$. This implies that there is y a $n+1$-chain of K' such that $\sum_{i=1}^{\beta_n} \varepsilon_i \chi(\mathrm{g}_i) = \partial' y$. As K' has no $n+1$-cell, we have $y = 0$, so $\sum_{i=1}^{\beta_n} \varepsilon_i \chi(\mathrm{g}_i) = 0$.

χ is injective (by construction) and $\{\mathrm{g}_i\}_{i=1}^{\beta_n}$ is linearly independent, so $\{\chi(\mathrm{g}_i)\}_{i=1}^{\beta_n}$ is linearly independent. Hence, $\forall i, \varepsilon_i = 0$, so $\left\{[\chi(\mathrm{g}_i)]^{K'}\right\}_{i=1}^{\beta_n}$ is linearly independent.

As a result, $\{\chi(\mathrm{g}_i)\}_{i=1}^{\beta_n}$ is a homology basis.

It remains to prove that $\{\chi(\mathrm{g}_i)\}_{i=1}^{\beta_n}$ is explicit. For simplicity of notation, we denote $\chi_i := \chi(\mathrm{g}_i)$.

First, we prove that every $\overline{\chi_j}$ has an element that is not in the other $\overline{\chi_i}$.

Recall the following property, implied by the fact that $\chi_i = \mathrm{g}_i - \langle \mathrm{g}_i, \tau \rangle (\tau - t)$: for every i, we have either $\tau \notin \overline{\mathrm{g}_i}$ and $\overline{\chi_i} = \overline{\mathrm{g}_i}$, or $\tau \notin \overline{\mathrm{g}_i}$ and $\overline{\chi_i} = (\overline{\mathrm{g}_i} \backslash \{\tau\}) \sqcup T_n$.

$\{\mathrm{g}_i\}_{i=1}^{\beta_n}$ is explicit, hence, fix j and choose $\tau_j \in \overline{\mathrm{g}_j} \backslash \cup_{i \neq j} \overline{\mathrm{g}_i}$. There are two cases:

- if $\tau_j \neq \tau$, then τ_j is in $\overline{\mathrm{g}_j} \backslash \{\tau\} \subseteq \overline{\chi_j}$. Moreover, τ_j is neither in T_n nor in $\overline{\mathrm{g}_i}$ (for $i \neq j$) so we have $\tau_j \in \overline{\chi_j} \backslash \cup_{i \neq j} \overline{\chi_i}$.
- if $\tau_j = \tau$, then it means that τ is not in every other $\overline{\mathrm{g}_i}$. Hence, we have $\overline{\chi_j} = (\overline{\mathrm{g}_j} \backslash \{\tau\}) \sqcup T_n$ and $\overline{\chi_i} = \overline{\mathrm{g}_i}$ for $i \neq j$. Let τ' be a cell of T_n. Then we have $\tau' \in \overline{\chi_j} \backslash \cup_{i \neq j} \overline{\chi_i}$.

Second, we prove the second property of explicit bases. Consider K'^{χ} the geometric subcomplex of K' generated by the cells of $\{\chi_i\}_{i=1}^{\beta_n}$ and all their faces[6]. We show that $\beta_n\left(K'^{\chi}\right) = \beta_n$.

To prove that, consider K^{g} the geometric subcomplex of K generated by the cells of $\{\mathrm{g}_i\}_{i=1}^{\beta_n}$ and their faces[6]. We show that $H_n(K'^{\chi}) \approx H_n(K^{\mathrm{g}})$.

Denote G and X the set of cells of K^{g} and K'^{χ}, respectively. In particular, we have $G_n := \cup_{i=1}^{\beta_n} \overline{\mathrm{g}_i}$ and $X_n := \cup_{i=1}^{\beta_n} \overline{\chi_i}$. Moreover, concerning their geometric realization we have $|K^{\mathrm{g}}| = |G|$ and $|K'^{\chi}| = |X|$. There are two cases:

- $\tau \notin G_n$: then it means that for all i, $\langle \mathrm{g}_i, \tau \rangle = 0$. Hence, for all i we have $\mathrm{g}_i = \chi_i$, therefore $K^{\mathrm{g}} = K'^{\chi}$. Hence, $H_n(K'^{\chi}) = H_n(K^{\mathrm{g}})$.
- $\tau \in G_n$: in this case, let j such that $\langle \mathrm{g}_j, \tau \rangle \neq 0$, so $\chi_j = \mathrm{g}_j - \tau + t$. As a consequence, we have $\overline{\chi_j} = (\overline{\mathrm{g}_j} \backslash \{\tau\}) \sqcup T_n$. Hence, this yields $X_n = (G_n \backslash \{\tau\}) \sqcup T_n$. By closing under the face operator, and because the faces of τ are contained in the faces of cells in T_n, we obtain $X = (G \backslash \{\tau\}) \sqcup T$. As a result:

$$\begin{aligned} |K'^{\chi}| = |X| &= |(G \backslash \{\tau\}) \sqcup T| = |G \backslash \{\tau\}| \sqcup |T| \\ &= |G \backslash \{\tau\}| \sqcup |\tau| \qquad \text{by Definition 4.3.3} \\ &= |G| = |K^{\mathrm{g}}| \end{aligned}$$

K^{g} and K'^{χ} are a geometric complexes, we have

$$H_n(K'^{\chi}) \approx H_n(|K'^{\chi}|) = H_n(|K^{\mathrm{g}}|) \approx H_n(K^{\mathrm{g}})$$

$\{\mathtt{g}_i\}_{i=1}^{\beta_n}$ is explicit so $H_n(K^{\mathtt{g}})$ has dimension β_n. As $H_n(K'^{\chi}) \approx H_n(K^{\mathtt{g}})$, we finally get $\beta_n\left(K'^{\chi}\right) = \beta_n$. □

Once $\tilde{\chi}$ is well-defined, we can define $[\tilde{\chi}]$ using Lemma 4.3.7. Then, the injectivity of $[\tilde{\chi}]$ is a direct consequence of the injectivity of χ.

Proof. [*Proof of Proposition 4.3.12*] Lemma 4.3.7 states that there is one and only one explicit basis per n-configuration. Thus, the following function, that associates to every n-configuration of K its corresponding n-configuration of K', following $\tilde{\chi}$, is well-defined:

$$\begin{aligned}[\tilde{\chi}]:\ \mathbf{Conf}_n(K) &\to \mathbf{Conf}_n(K')\\ \left[\{\mathtt{g}_i\}_{i=1}^{\beta_n}\right] &\mapsto \left[\tilde{\chi}\left(\{\mathtt{g}_i\}_{i=1}^{\beta_n}\right)\right]\end{aligned}$$

Next, $\tilde{\chi}$ is injective. Indeed, $\tilde{\chi}\left(\{\mathtt{g}_i\}_{i=1}^{\beta_n}\right) = \tilde{\chi}\left(\{\mathtt{f}_i\}_{i=1}^{\beta_n}\right)$ implies that there is a permutation of indices ϕ such that $\chi(\mathtt{g}_i) = \chi(\mathtt{f}_{\phi(i)})$. As χ is injective, we have that $\mathtt{g}_i = \mathtt{f}_{\phi(i)}$, so $\{\mathtt{g}_i\}_{i=1}^{\beta_n} = \{\mathtt{f}_i\}_{i=1}^{\beta_n}$. It follows that $[\tilde{\chi}]$ is injective as well. □

4.3.4.4. Building an injection from $\mathbf{Conf}_n(K')$ to $\mathbf{Conf}_n(K)$

Recall that $(K',\partial) = (K^- \oplus \operatorname{Span} T, \partial)$ is an elementary subdivision of $(K,\partial) = (K^- \oplus \operatorname{Span}\{\tau\},\partial)$. Recall also that a n-tri-partition does not have any primary cell, as there are no $(n+1)$-cells. We define the following function:

$$\begin{aligned}p:\ \mathbf{TP}_n(K') &\to \mathbf{TP}_n(K)\\ (\emptyset, S, C) &\mapsto (\emptyset, \hat{S}, \hat{C})\end{aligned}$$

$$\text{With: } \begin{cases}\text{if } T\cap C = \{\kappa\}: & \hat{S} := S\cap\overline{K^-} & \hat{C} := (C\backslash\{\kappa\})\sqcup\{\tau\}\\ \text{if } T\cap C = \emptyset: & \hat{S} := (S\cap\overline{K^-})\sqcup\{\tau\} & \hat{C} := C\end{cases}$$

We want to prove the following proposition.

Proposition 4.3.14. *Let $[p]$ be the function that associates to every n-configuration of K' its corresponding n-configuration of K, following p. $[p]$ is well-defined, and is an injection from $\mathbf{Conf}_n(K')$ to $\mathbf{Conf}_n(K)$*

The idea is that p transforms a tri-partition on K' into a tri-partition on K while preserving the homology configuration. See Figure 4.22 for an illustration of p.

First, we show that p is well-defined, see Proposition 4.3.18. The proof of this proposition uses Corollary 4.3.17, which shows that the two cases presented in the definition of p are exhaustive. Corollary 4.3.17 requires the following lemmas about Betti numbers in subdivisions.

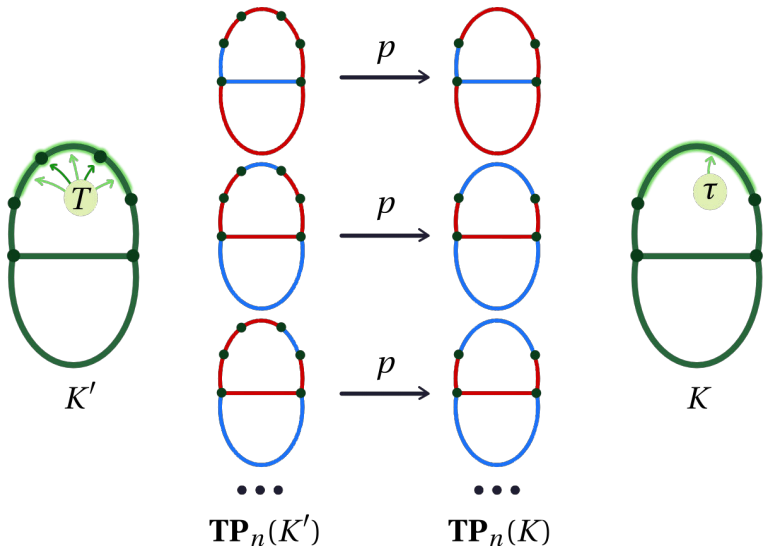


Figure 4.22 – Illustration of the function p.

Lemma 4.3.15. *Let (K^-,∂^-) be a complex and (K,∂) be the same complex with an added q-cell τ. In particular, $K = K^- \oplus \mathrm{Span}\{\tau\}$ and ∂ coincides with ∂^- on K^-. Then, we have $\beta_q(K) \leq \beta_q(K^-)+1$.*

Proof. Let $X^- = (P,S,C)$ be a perfect HDVF for (K^-,∂^-). $X := (P,S,C \sqcup \{\tau\})$ is a HDVF for (K,∂), as both boundary operators coincide on P and S. Hence, we have $\beta_q(K) \leq |C_q \sqcup \{\tau\}| = \beta_q(K^-)+1$. □

Lemma 4.3.16. *Let (K^-,∂^-) be a complex and (K,∂) be the same complex with an additional set of cells T of dimensions lesser or equal to $n = \dim(K^-)$. In particular, $K = K^- \oplus \mathrm{Span}\, T$ and ∂ coincides with ∂^- on K^-. Let $X = (P,S,C)$ be a perfect HDVF for (K,∂). Then we have $\beta_n(K^-) \leq \beta_n(K) - |T \cap C_n|$.*

Proof. Let (f,g,h) be the reduction associated to X. $\mathtt{j}_S h \mathtt{i}_P$ is invertible so there exists an invertible submatrix $\mathtt{j}_{T\cap S} h \mathtt{i}_\Pi$ (with $\Pi \subseteq P$). Hence, $\mathtt{R}_\Pi^{T\cap S}$ [7] is a valid operation for X. We define $X' := \mathtt{R}_\Pi^{T\cap S} X = (P',S',C')$ which is a HDVF for (K,∂). Notably, we have $T_n \subseteq C'_n$ and $C'_n = C_n \sqcup (T \cap S_n)$ (where T_n denote the n-cells of T).

Denote $(\tilde{K},\partial) := (K^- \oplus \mathrm{Span}\,(T \setminus T_n),\partial)$ the subcomplex of K where the n-cells of T are removed. $(\tilde{K},\partial)$ is a well-defined complex as n is the maximal dimension of K. Define $X^- = (P^-,S^-,C^-)$ with:

$$P^- = P' \qquad S^- = S' \qquad C^- = C' \setminus T$$

In particular, we have $C_n^- = C_n \setminus (T \cap C_n)$

X^- is a HDVF for the complex (K^-,∂). Indeed, it is a partition of the cells of K^- because $T_n \subseteq C'_n$ and (P',S',C') is a partition of K. Moreover, as X' is a HDVF, we have $\partial_{S^-}^{P^-} = \partial_{S'}^{P'}$ is invertible. Finally, this gives a bound for the n-th Betti number of

$\tilde{K}$:

$$\begin{aligned}\beta_n(\tilde{K}) \leq |C_n^-| &= |C_n \setminus (T \cap C_n)| = |C_n| - |T \cap C_n| \\ &\leq \beta_n(K) - |T \cap C_n| \qquad \text{because } X \text{ is perfect}\end{aligned}$$

The proof is concluded by using the fact that $\beta_n(\tilde{K}) = \beta_n(K^-)$. Indeed, $\tilde{K}$ and K^- have the same n-cells with the same boundary matrix: $\partial \mathtt{i}_{\tilde{K}_n} = \partial \mathtt{i}_{K_n^-}$. Thus, they have isomorphic n-homology groups. □

Now, recall our setting: $(K', \partial) = (K^- \oplus \operatorname{Span} T, \partial)$ is an elementary subdivision of $(K, \partial) = (K^- \oplus \operatorname{Span}\{\tau\}, \partial)$, with $\dim \tau = n$. The two previous lemmas imply:

Corollary 4.3.17. *Let $X = (P, S, C)$ be a perfect HDVF of (K', ∂). Then $|T \cap C_n| \leq 1$ i.e. there is at most one critical n-cell in T.*

Proof. Using Lemma 4.3.15: $\beta_n(K) \leq \beta_n(K^-) + 1$. Moreover, by Lemma 4.3.16, $\beta_n(K^-) \leq \beta_n(K') - |T \cap C_n|$. Finally, as $(K, \partial) \prec (K', \partial)$, we have $\beta_n(K) = \beta_n(K')$. Combining these results, we get $|T \cap C_n| \leq \beta_n(K) - \beta_n(K^-) \leq 1$. □

Once we have Corollary 4.3.17, we can prove that p is well-defined.

Proposition 4.3.18. *p is well-defined i.e. given X a n-tri-partition for K', $p(X)$ is a n-tri-partition for K.*

Proof. Let $X = (\emptyset, S, C)$ be a n-tri-partition for (K', ∂) and denote $\hat{X} := p(X) = (\emptyset, \hat{S}, \hat{C})$. To begin with, $\hat{X}$ is a partition of $\overline{K_n}$ because $\overline{K_n} = \overline{K^-} \sqcup \{\tau\} = \overline{K'_n} \setminus T \sqcup \{\tau\}$, and because $(\emptyset, S, C)$ is a partition of $\overline{K'_n}$. In addition, Proposition 2.2.5 and Corollary 4.3.17 implies that the two cases in the definition of p are exhaustive.

We now show that $\hat{X}$ is a tri-partition. Please refer to Figure 4.22 to understand the function p.

- **Case $T \cap C = \{\kappa\}$.** $\hat{S} = S \cap \overline{K^-}$ so $\hat{S}$ is a tree because S is a tree. We have:

$$\begin{aligned}|\hat{C}| &= |(C \setminus \{\kappa\}) \sqcup \{\tau\}| = |C| \\ &= \beta_n(K') && \text{as } (\emptyset, S, C) \text{ is a } n\text{-tri-partition of } K' \\ &= \beta_n(K) && \text{as } K' \text{ is a subdivision of } K\end{aligned}$$

  By cardinality, S is a maximal tree and $\hat{X}$ is a tri-partition of K.

- **Case $T \cap C = \emptyset$.** we first show that $\hat{S}$ is a tree. Let x be a cycle in $\operatorname{Span} \hat{S} = \operatorname{Span}(S \cap \overline{K^-}) \oplus \operatorname{Span}\{\tau\}$. There exists $\lambda \in \mathscr{R}$ and $s \in \operatorname{Span}(S \cap \overline{K^-})$ such that $x = \lambda\tau + s$.

  As x is a cycle, we have that $\chi(x) = \chi(s) + \chi(\tau) = s + t$ is a cycle as well. $s \in \operatorname{Span} S$, moreover, as $T \cap C = \emptyset$, we have that $T \subseteq S$ so $t \in \operatorname{Span} S$. As a result, $\chi(x)$ is

[7] $\mathtt{R}$ is the opposite operation of $\mathtt{A}$, see GONZALEZ-LORENZO 2016 for more information about it.

a cycle in K' composed of cells of S, so it is equal to 0 as S is a tree of K'. By injectivity of χ, $x = 0$.

All in all, $\hat{S}$ is a tree, moreover, $|\hat{C}| = |C| = \beta_n(K') = \beta_n(K)$ so $\hat{X}$ is a tri-partition of K by cardinality. □

Finally, we are ready to prove Proposition 4.3.14, which states that $p : \mathbf{TP}_n(K') \to TP_n(K)$ can be extended to $[p] : \mathbf{Conf}_n(K') \to \mathbf{Conf}_n(K)$, which is injective. Precisely, we first show Lemma 4.3.19, which relates the canonical cycles of X and $p(X)$. Then, we combine this lemma with Lemma 4.3.7, to prove that $X \sim'_n X'$ (in (K', ∂')) if and only if $p(X) \sim_n p(X')$ (in (K, ∂)), which is enough to conclude the proof of Proposition 4.3.14.

Lemma 4.3.19. *Let $X = (\emptyset, S, C)$ be a n-tri-partition of (K', ∂) and let $\hat{X} = (\emptyset, \hat{S}, \hat{C}) := p(X)$ be its image under p. Let z (resp. $\hat{z}$) be the canonical cycle map of X (resp. $\hat{X}$). Then there exist a bijection $\psi : \hat{C} \to C$ such that*

$$\forall \gamma \in \hat{C}, \quad \chi \circ \hat{z}(\gamma) = z \circ \psi(\gamma)$$

Precisely, we can choose ψ as follows.

- *If $T \cap C = \emptyset$, then $\hat{C} = C$ and $\psi := id$.*
- *If $T \cap C = \{\kappa\}$, then $\hat{C} = (C \setminus \{\kappa\}) \sqcup \{\tau\}$. We set $\psi(\tau) = \kappa$ and $\psi(\sigma) = \sigma$ for $\sigma \neq \tau$.*

Proof. We consider the two following cases:

- **Case $T \cap C = \emptyset$.** $\hat{C} = C \subseteq \overline{K^-}$ and $\hat{S} = (S \cap \overline{K^-}) \sqcup \{\tau\}$.
  Let $\gamma \in \hat{C}$. Following Lemma 1.6.3, we decompose the cycle $\hat{z}(\gamma)$ in three parts: $\hat{z}(\gamma) = \gamma + \langle \hat{z}(\gamma), \tau \rangle \tau + \hat{s}$, with $\hat{s} \in \mathrm{Span}\,(\hat{S} \setminus \{\tau\})$. Hence, when applying χ, we get:

  $$\begin{aligned} \chi\hat{z}(\gamma) &= \chi(\gamma) + \langle \hat{z}(\gamma), \tau \rangle t + \chi(\hat{s}) \\ &= \gamma + \langle \hat{z}(\gamma), \tau \rangle t + \chi(\hat{s}) && \text{as } \gamma \in hatC \subseteq \overline{K^-} \text{ so } \gamma \neq \tau \\ &= \gamma + \langle \hat{z}(\gamma), \tau \rangle t + \hat{s} && \text{as } \hat{S} \setminus \{\tau\} \subseteq \overline{K^-} \text{ so } \tau \notin \bar{\hat{s}} \end{aligned}$$

  As $T \cap C = \emptyset$, we have $t \in \mathrm{Span}\, S$. Moreover, $\hat{s} \in \mathrm{Span}\,(\hat{S} \setminus \{\tau\}) \subseteq S$. Thus, $\chi\hat{z}(\gamma)$ is a cycle composed of $\gamma \in C$ plus chains of $\mathrm{Span}\, S$. By Lemma 1.6.3, $\chi\hat{z}(\gamma) = z(\gamma)$.
- **Case $T \cap C = \{\kappa\}$.** $\hat{C} = C \setminus \{\kappa\} \sqcup \{\tau\}$ and $\hat{S} = S \cap \overline{K^-}$.
  Let $\gamma \in \hat{C}$. Following Lemma 1.6.3, we decompose the cycle $\hat{z}(\gamma)$: $\hat{z}(\gamma) = \gamma + \hat{s}$, with $\hat{s} \in \mathrm{Span}\, \hat{S} \subseteq \mathrm{Span}\, S$.

  First, consider the case $\gamma \in \hat{C} \setminus \{\tau\}$. In this case, both γ and $\hat{s}$ are in K^-. Hence, we obtain:

  $$\chi\hat{z}(\gamma) = \chi(\gamma) + \chi(\hat{s}) = \gamma + \hat{s}$$

  Thus, $\chi\hat{z}(\gamma)$ is a cycle composed of $\gamma \in C$ plus a chain of $\hat{s} \in \mathrm{Span}\, S$. By Lemma 1.6.3, $\chi\hat{z}(\gamma) = z(\gamma)$.

Second, consider $\gamma = \tau$. $T \cap C = \{\kappa\}$ so we have that $t = \sum_{\sigma \in T_n \sigma} = \kappa + t'$, with $t' \in \text{Span}\, T \cap \text{Span}\, S$. We get:

$$\begin{aligned} \chi\hat{z}(\tau) &= \chi(\tau) + \chi(\hat{s}) = t + \hat{s} \\ &= \kappa + t' + \hat{s} \end{aligned}$$

Thus, $\chi\hat{z}(\tau)$ is a cycle composed of $\kappa \in C$ plus the chains of t' and $\hat{s}$ that are in $\text{Span}\, S$. By Lemma 1.6.3, $\chi\hat{z}(\tau) = z(\kappa)$. □

Proof. [*Proof of Proposition 4.3.14*] First, we prove that if X^0 and X^1 are two tri-partitions for (K', ∂'), then, $X^0 \sim'_n X^1$ if and only if $p(X^0) \sim_n p(X^1)$.

Let $X^0 = (\emptyset, S^0, C^0)$ and $X^1 = (\emptyset, S^1, C^1)$ be two tri-partitions for (K', ∂'), with canonical cycle maps z^0 and z^1 respectively. Denote $\hat{X}^0 = (\emptyset, \hat{S}^0, \hat{C}^0) := p(X^0)$ and $\hat{X}^1 = (\emptyset, \hat{S}^1, \hat{C}^1) := p(X^1)$ tri-partitions for (K, ∂) (by Proposition 4.3.18), with canonical cycle maps $\hat{z}^0$ and $\hat{z}^1$ respectively.

Denote $\psi_0 : \hat{C}^0 \to C^0$ and $\psi_1 : \hat{C}^1 \to C^1$ the bijections given by applying Lemma 4.3.19 to X^0 and X^1.

We can now prove $X^0 \sim'_n X^1 \iff \hat{X}^0 \sim_n \hat{X}^1$. In the following equivalences, we use the notation "$\phi : C \leftrightarrow C'$" to mean that ϕ is a bijection from C to C'.

$$\begin{aligned} X^0 \sim'_n X^1 &\iff \exists \phi : C^0 \leftrightarrow C^1,\ \forall \gamma \in C^0, \quad [z^0(\gamma)]^{K'} = [z^1\phi(\gamma)]^{K'} \\ &\iff \exists \phi : C^0 \leftrightarrow C^1,\ \forall \gamma \in C^0, \quad z^0(\gamma) = z^1\phi(\gamma) && \text{by Lemma 4.3.7} \\ &\iff \exists \phi : C^0 \leftrightarrow C^1,\ \forall \gamma \in C^0, \quad \chi\hat{z}^0\psi_0(\gamma) = \chi\hat{z}^1\psi_1\phi(\gamma) && \text{by Lemma 4.3.19} \\ &\iff \exists \phi : C^0 \leftrightarrow C^1,\ \forall \gamma \in C^0, \quad \hat{z}^0\psi_0^{-1}(\gamma) = \hat{z}^1\psi_1^{-1}\phi(\gamma) && \text{as } \chi \text{ is injective} \\ &\iff \exists \hat{\phi} : \hat{C}^0 \leftrightarrow \hat{C}^1,\ \forall \hat{\gamma} \in \hat{C}^0, \quad \hat{z}^0(\hat{\gamma}) = \hat{z}^1\hat{\phi}(\hat{\gamma}) && \hat{\phi} = \psi_1^{-1}\phi\psi_0 \\ &\iff \exists \hat{\phi} : \hat{C}^0 \leftrightarrow \hat{C}^1,\ \forall \hat{\gamma} \in \hat{C}^0, \quad [\hat{z}^0(\hat{\gamma})]^K = [\hat{z}^1\hat{\phi}(\hat{\gamma})]^K && \text{by Lemma 4.3.7} \\ &\iff \hat{X}^0 \sim_n \hat{X}^1 \end{aligned}$$

In particular, $X \sim'_n X' \implies p(X) \sim_n p(X')$. Thus, the following function, that associates to every n-configuration of K' its corresponding n-configuration of K, is well-defined:

$$\begin{aligned} [p] : \mathbf{Conf}_n(K') &\to \mathbf{Conf}_n(K) \\ [X] &\mapsto [p(X)] \end{aligned}$$

Moreover, as $p(X) \sim_n p(X') \implies X \sim'_n X'$, this function is injective. □

All in all, Propositions 4.3.12 and 4.3.14 prove Theorem 4.3.10, which we restate here:

Theorem (combinatorial invariance of homology configurations)**.** *Let (K, ∂) and (K', ∂') be two geometric n-complexes such that $K \prec K'$. Then,* $\mathbf{Conf}_n(K) \approx \mathbf{Conf}_n(K')$.

4.3.5. Topological invariance and the Hauptvermutung

The *Hauptvermutung* (German for "main conjecture") is a central question in geometric topology stated in the early 20th century. It asks whether two triangulations of the same topological space have a common subdivision.

The conjecture comes from the work of Steinitz and Tietze in 1908 (STEINITZ 1908; TIETZE 1908), and was believed to be true for several decades. It holds in dimension 1 and was proved in dimension 2 by Moise in the 1950s (MOISE 1952), who also established that every 2-manifold admit a unique piecewise-linear (PL) structure up to homeomorphism.

Moise also proved the conjecture for 3-manifolds, showing that every 3-manifold admits a unique PL structure MOISE 1952. However, in higher dimensions, the situation is different.

A breakthrough came in the 1960s with work by Milnor and others, revealing that in dimension ≥ 5, the Hauptvermutung fails in general (MILNOR 1961). Counterexamples were constructed using exotic spheres and the study of triangulations incompatible with any PL structure.

Today, the Hauptvermutung is known to be false in general for dimensions ≥ 4 (see for instance FREEDMAN et al. 1990), but remains true in lower dimensions. In particular, BROWN 1969 proved that it is true for every simplicial complex of dimension ≤ 3.

Theorem 4.3.20 (Hauptvermutung, BROWN 1969)**.** *Let K and K' be two n-dimensional simplicial complexes (with $n \leq 3$) such that $|K|$ and $|K'|$ are homeomorphic. Then, there exist two simplicial complexes $\hat{K}$ and $\hat{K}'$ such that:*

- $K \prec \hat{K}$.
- $K' \prec \hat{K}'$.
- *$\hat{K}$ and $\hat{K}'$ are* combinatorially equivalent, *i.e. there exists a bijection between cells of $\hat{K}$ and $\hat{K}'$ that induces a chain isomorphism between the combinatorial chain complexes of $\hat{K}$ and $\hat{K}'$.*

This theorem implies that for dimension 3 or lower, every combinatorial invariant is a topological invariant. In particular, this applies to homology configurations of maximal dimension, as presented in the following theorem.

Theorem 4.3.21. *The number of homology configurations in the maximal dimension is a topological invariant for simplicial complexes of dimension lower than* 3.

Precisely, if K and K' are two n-dimensional simplicial complexes (with $n \leq 3$) such that $|K|$ and $|K'|$ are homeomorphic, then $\mathbf{Conf}_n(K) \approx \mathbf{Conf}_n(K')$. See Table 4.1 for a classification of some low-dimensional complexes derived from $\mathbf{Conf}_n(K)$.

Proof. Let K and K' be two n-dimensional cellular complexes (with $n \leq 3$) such that $|K|$ and $|K'|$ are homeomorphic. Theorem 4.3.20 implies that there exist $\hat{K}$ and $\hat{K}'$ that are combinatorially equivalent and such that $K \prec \hat{K}$ and $K' \prec \hat{K}'$. As $\hat{K}$ and $\hat{K}'$ have isomorphic combinatorial chain complexes with isomorphic notion of cells,

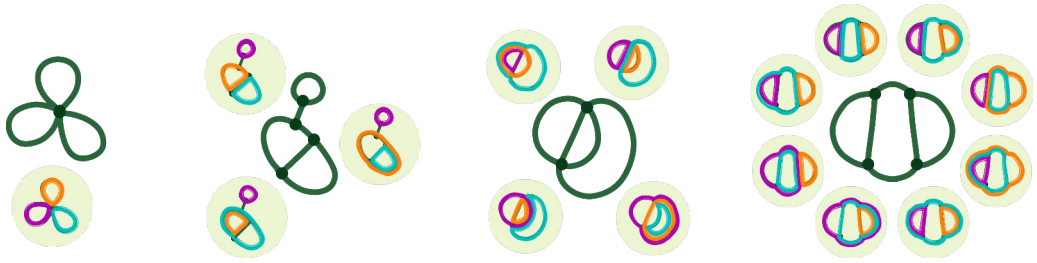
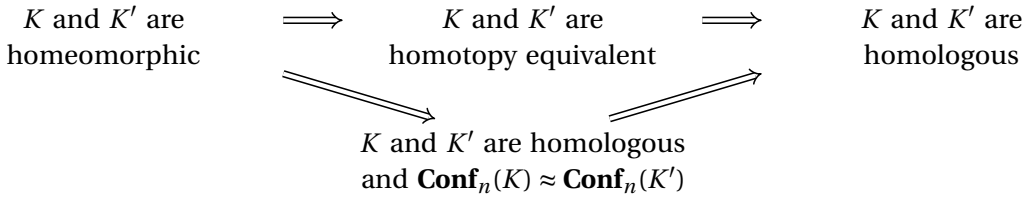


Figure 4.23 – 1-configurations of four 1-complexes. All these complexes are homotopy equivalent, but have different numbers of 1-configurations, illustrating the fact that homology configurations can sometimes be more discriminative than homotopy. Intuitively, the number of configurations seems to measure how much holes are related. A small number of configurations indicate that holes are somehow "independent", whereas a large number of configurations means that holes are somehow "co-lateral".

we have $\mathbf{Conf}_n(\hat{K}) \approx \mathbf{Conf}_n(\hat{K}')$. By Theorem 4.3.10, we have $\mathbf{Conf}_n(K) \approx \mathbf{Conf}_n(\hat{K})$ and $\mathbf{Conf}_n(K') \approx \mathbf{Conf}_n(\hat{K}')$.

All in all we obtain $\mathbf{Conf}_n(K) \approx \mathbf{Conf}_n(K')$. □

Homology configurations do not fit in the traditional hierarchy of topological invariants. Indeed, although they come from homology, they can discriminate shapes that are homotopy equivalent, see for instance Figure 4.23. On the contrary, there exist shapes that are not homotopy equivalent, although they have isomorphic homology groups and the same number of configurations. One such example is a torus compared to the wedge sum of a spheres with two circles (sometimes called a "Mickey"). Both shapes have isomorphic homology groups and possess a single 2-configuration, but they are not homotopy equivalent.



The study of homology configurations raises the broader question of finding and understanding combinatorial invariants of cellular complexes, that is, properties that remain unchanged under subdivisions of the complex. For instance, one may consider as a related combinatorial invariant the number of elementary n-cycles. Other potential examples may arise from graph theory, in the same manner as explicit bases generalize strictly fundamental cycle bases for graphs. Such quantities constitute natural candidates for topological invariants. Indeed, for complexes of dimension at most 3, they are topological invariants as a consequence of the Hauptvermutung.

This leads to the following open questions.

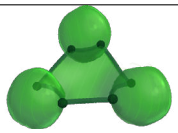
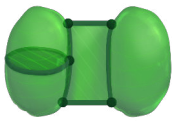
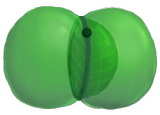
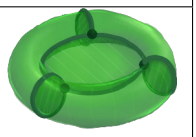
	$\lvert\mathbf{Conf}_n(K)\rvert$ (with $n = \dim K$)				
	1	3	4	8	16
$n=1$					
$n=2$					

Table 4.1 – Classification of some CW-complexes K of dimension $n = 1$ or $n = 2$, that have exactly three n-holes. The complexes are classified depending on the cardinal of $\mathbf{Conf}_n(K)$. All the complexes of the $n = 1$ row are homotopy equivalent, but have different numbers of 1-configurations, so they are not homeomorphic. Similarly, the complexes of the $n = 2$ row are homotopy equivalent, but have different numbers of 2-configurations, so they are not homeomorphic.

Question 2. *Does* $\mathbf{Conf}_n(K)$ *provide a topological invariant for complexes of dimension* $n \geq 4$*?*

Question 3. *Can we derive a combinatorial or topological invariant from the homology configurations of arbitrary dimensions?*

Recall Figure 4.18 which illustrates that for general dimension q, $\mathbf{Conf}_q(K)$ is not preserved under subdivision. Despite this observation, we have the following conjecture:

Conjecture 4.3.22. *Subdividing a complex always increases its number of* q*-configurations. In addition, after* β_q *barycentric subdivisions, the number of* q*-configurations cannot increase further.*

Other general open questions include:

Question 4. *How can homology configurations be defined in the setting of singular homology? In particular, for a geometric complex* K*, do we have* $\mathbf{Conf}_q(K) \approx \mathbf{Conf}_q(|K|)$*?*

Question 5. *Can homology configurations be used to study knots and links?*

A potential interesting case would be to compute configurations of *bonded* links and their complementary. See for instance Diamantis et al. 2025 for an introduction to *bonded* knots and links.

4.3.6. Computing homology configurations

In this part, we discuss the computation of homology configurations. In particular, we present the graph of n-configurations, and give some bounds for the complexity of its computation.

The graph of homology configurations in maximal dimension

To begin with, we prove two lemmas that shows that $\mathbf{Conf}_n(K)$ can be equipped with a graph structure.

The following lemma states that every n-configuration defines a connected subgraph of the graph of perfect HDVFs (whose vertices are HDVFs and whose edges correspond to operations).

Lemma 4.3.23. *Given two perfect HDVFs in the same n-configuration, there exists a sequence of valid* W *operations that transform the first HDVF into the other while staying in the configuration.*

Proof. Let $X \sim_n X'$ be two perfect HDVFs for K. Every critical n-cell γ of X is associated with a critical n-cell γ' of X', such that $[g(\gamma)]_n = [g'(\gamma')]_n$, which implies $g(\gamma) = g'(\gamma')$ by Lemma 4.3.7.

Whenever $\gamma' \neq \gamma$, we can transform X with $\mathtt{W}^{\gamma}_{\gamma'}$ operations until we obtain X'. Every $\mathtt{W}^{\gamma}_{\gamma'}$ is a valid operation as $\langle g(\gamma), \gamma' \rangle \neq 0$. We now prove that these $\mathtt{W}^{\gamma}_{\gamma'}$ operations preserve the configuration.

We have $g^*(\gamma') = \sum_{\tau \in C} \langle g(\tau), \gamma' \rangle \tau$. However, γ is the only cell of C such that $\langle g(\tau), \gamma' \rangle \neq 0$, otherwise this would contradict that fact that $f'g = \mathtt{j}_{C'}g$ is a permutation matrix.

Hence, we obtain $g^*(\gamma' - \gamma) = g^*(\gamma) - g^*(\gamma) = 0$. By Proposition 4.3.5, every $\mathtt{W}^{\gamma}_{\gamma'}$ operation transforms the HDVF without changing the configuration. □

The second lemma shows that W is actually well-defined over configurations: if a W operation send a HDVF X to a n-configuration α, then any HDVF $X' \sim_n X$ can be sent to α by a W operation.

Lemma 4.3.24. *Let X and X' be two HDVFs such that $X \sim_n X'$. For every valid operation $\mathtt{W}^{\gamma}_{\sigma}$ for X that modifies the configuration, there exists a valid operation $\mathtt{W}^{\gamma'}_{\sigma'}$ for X' such that $\mathtt{W}^{\gamma}_{\sigma} X \sim_n \mathtt{W}^{\gamma'}_{\sigma'} X'$.*

Proof. Let $X = (P, S, C) \sim_n X' = (P', S', C')$ be two perfect HDVFs for K. Denote $\phi = f'g$ the permutation from the cells of X to the cells of X'. By Lemma 4.3.7, for every n-cell $\tau \in C$ we have $g(\tau) = g'(\phi\tau)$. Let $\gamma' = \phi\gamma$, which implies $g(\gamma) = g'(\gamma')$.

$\mathtt{W}^{\gamma}_{\sigma}$ is valid for X so $\sigma \in S$ and $\langle g(\gamma), \sigma \rangle \neq 0$.

Because $g'(\gamma') = g(\gamma)$, we have $\langle g'(\gamma'), \sigma \rangle \neq 0$. In addition, $\sigma \in S'$. Indeed, otherwise σ would be in C'. This would imply $\sigma = \gamma'$, because $\sigma \in \overline{g'(\gamma')}$ and γ' is the only cell of C' in $g'(\gamma')$. In this case we would obtain $g * (\sigma - \gamma) = 0$, which would contradict the fact that $\mathtt{W}^{\gamma}_{\sigma}$ modifies the configuration of X (by Proposition 4.3.5).

As a result, $\sigma \in S'$ and $\langle g'(\gamma'), \sigma \rangle \neq 0$, so $\mathtt{W}^{\gamma'}_{\sigma}$ is a valid operation for X'.

Denote $\hat{X} = \mathtt{W}^{\gamma}_{\sigma} X$ and $\hat{X}' = \mathtt{W}^{\gamma'}_{\sigma} X'$. We now prove that these two HDVFs induce the same n-homology basis.

By Lemma 1.6.8, we have $\hat{g}(x) = z(x) - \xi \langle z(x) - x, \sigma \rangle g(\gamma)$, with $\xi = \langle \hat{g}(\gamma), \sigma \rangle^{-1}$. Similarly, $\hat{g}'(x) = z'(x) - \xi \langle z'(x) - x, \sigma \rangle g'(\gamma')$. Note that the coefficient ξ is the same

in both expressions because $g(\gamma) = g'(\gamma)$. In addition, we have $\hat{C} = (C \setminus \{\gamma\}) \sqcup \{\sigma\}$ and $\hat{C}' = (C' \setminus \{\gamma'\}) \sqcup \{\sigma\}$.

We obtain the following equalities between the generators induced by $\hat{X}$ and $\hat{X}'$. Let $\tau \in C \setminus \{\gamma\}$ and $\phi\tau \in C' \setminus \{\gamma'\}$ be its corresponding cell, then

$$\begin{aligned}
\hat{g}(\tau) &= g(\tau) - \xi\langle g(\tau) - \tau, \sigma\rangle g(\gamma) \\
&= g(\tau) - \xi\langle g(\tau), \sigma\rangle g(\gamma) && \text{as } \tau \in C \text{ but } \sigma \in S \\
\hat{g}'(\phi\tau) &= g'(\phi\tau) - \xi\langle g'(\phi\tau) - \phi\tau, \sigma\rangle g'(\gamma') \\
&= g(\tau) - \xi\langle g(\tau) - \phi\tau, \sigma\rangle g(\gamma) && \text{because } g'(\phi\tau) = g(\tau) \text{ and } g(\gamma) = g'(\gamma') \\
&= g(\tau) - \xi\langle g(\tau), \sigma\rangle g(\gamma) && \text{as } \phi\tau \in C' \text{ but } \sigma \in S'
\end{aligned}$$

Moreover,

$$\begin{aligned}
\hat{g}(\sigma) &= z(\sigma) - \xi\langle z(\sigma) - \sigma, \sigma\rangle g(\gamma) \\
&= g(\gamma) && \text{as } z(\sigma) = 0 \\
\hat{g}'(\sigma) &= z'(\sigma) - \xi\langle z'(\sigma) - \sigma, \sigma\rangle g'(\gamma') \\
&= g(\gamma) && \text{as } z'(\sigma) = 0 \text{ and } g(\gamma) = g'(\gamma')
\end{aligned}$$

Hence, $\{\hat{g}(\tau)\}_{\tau \in \hat{C}}$ is equal to $\{\hat{g}'(\tau)\}_{\tau \in \hat{C}'}$ so $\hat{X} \sim_n \hat{X}'$. □

All in all, Lemmas 4.3.23 and 4.3.24 implies that $\mathbf{Conf}_n(K)$ can be equipped with a graph structure. The vertices of this graph correspond to configurations, and two configurations A and B are connected by an edge if there exists a valid `W`-operation transforming a HDVF of A into a HDVF of B. Equivalently, by Lemma 4.3.23, two configurations A and B are connected if, for every HDVF X of A, there exists a valid `W`-operation transforming X into a HDVF of B.

In addition, by Theorem 3.3.1, this graph is connected. See Figure 4.24 for an example of such graph.

Computation and complexity of the graph of homology configurations

Using this graph, we can derive an algorithm for computing homology configurations in maximal dimension:

1. compute a perfect tri-partition X in maximal dimension n for given complex K.
2. Perform a breadth-first search on the graph of $\mathbf{Conf}_n(K)$, using `W` operations on the tri-partition:
   - use Proposition 4.3.5 to select `W` operations that change the configuration.
   - Store the configurations that have already been processed (for instance, by recording the critical cells of a representative tri-partition).
   - Use Lemma 4.3.8 to determine whether two tri-partitions belong to the same configuration, in order to avoid processing the same configuration multiple times.

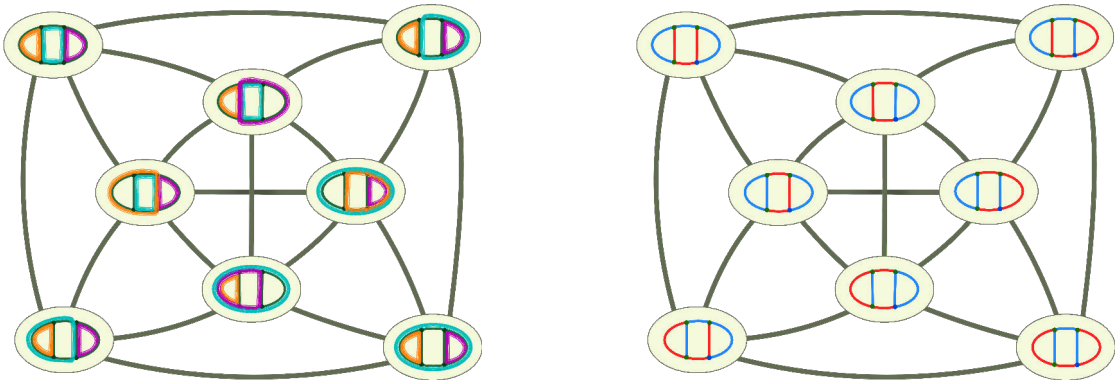


Figure 4.24 – The graph of the 1-configurations of a 1-dimensional complex, which has three 1-holes. On the left, each configuration is represented by an explicit homology basis. On the right, each configuration is represented with a representative HDVF.

The algorithm terminates, as the number of configurations is finite. Moreover, because the graph of configurations is connected, the algorithm enumerates all configurations. Its time complexity is $O(N^3 + N^2|\mathbf{Conf}_n(K)|)$, where N denotes the number of n-cells of K.

Since each n-homology configuration represents a unique explicit n-homology basis (by Lemma 4.3.7), this algorithm can also be used to compute minimal explicit n-homology bases.

In the case of complexes of dimension 1 (i.e. graphs), this yields an algorithm with complexity $O(N^3 + N^2|\mathbf{Conf}_n(K)|)$ to solve the problem of computing minimal strictly fundamental cycle bases, which is known to be NP-complete (Deo et al. 1982), even for planar graphs (Reich 2016). Thus, this prove that, in general, $|\mathbf{Conf}_n(K)|$ is not bounded by a polynomial in N (unless $P = NP$).

However, by definition, $\mathbf{Conf}_q(K)$ is bounded by the number of bases of $H_q(K)$. As we consider the field $\mathscr{R} = \mathbb{Z}/2\mathbb{Z}$, we can use the formula of the sequence A053601(OEIS Foundation Inc. 2026) counting the number of bases of $\mathbb{Z}/2\mathbb{Z}^{\beta_q}$, which gives:

$$|\mathbf{Conf}_q(K)| \leq \frac{1}{\beta_q!} \prod_{i=0}^{\beta_q - 1} \left(2^{\beta_q} - 2^i\right)$$

This highlights that the number of q-configurations mostly depends on β_q rather than N. This bound also raises the following question:

Question 6. *What are the complexes that produce the most q-configurations, given a fixed β_q?*

Looking at Figure 4.23, an intuition is that the more "co-lateral" n-holes are, the more numerous the n-configurations are.

We present other open questions and conjectures related to the graph structure of homology configurations. First, as the set $\mathbf{Conf}_n(K)$ is a combinatorial invariant, it is

natural to wonder if it is still a combinatorial invariant when equipped with its graph structures. This could potentially provide a more accurate topological invariant.

Conjecture 4.3.25. *Let $K \prec K'$ be n-dimensional complexes. Then, the graph associated to* $\mathbf{Conf}_n(K)$ *is isomorphic to the graph associated to* $\mathbf{Conf}_n(K')$.

Regarding the computation of configuration, in lower dimensions, it is still possible to enumerate all homology configurations by exploring the entire graph of perfect HDVFs. However, this approach is not efficient. This motivates the following open question:

Question 7. *Can homology configurations in arbitrary dimension be equipped with a graph structure, similar to the one defined in maximal dimension?*

Finally, the graph of n-configurations appears to be related to certain classes of graphs, namely *flip graphs*. For instance, the graph of $\mathbf{Conf}_n(K)$ for the complexes with 16 configurations in Table 4.1 directly corresponds to the flip graph of spanning trees of the complete graphs with 4 vertices (for an illustration of this graph, see for instance Figure 1(g) in MÜTZE 2022).

Question 8. *How are the graphs of homology configurations related to flip graphs of spanning trees?*

Topological operations and configurations

To conclude this section, another way of computing homology configurations would be to understand how standard elementary topological operations modify homology configurations. We did not investigate this question yet, but we provide two conjectures, which are illustrated in Figures 4.25 and 4.26.

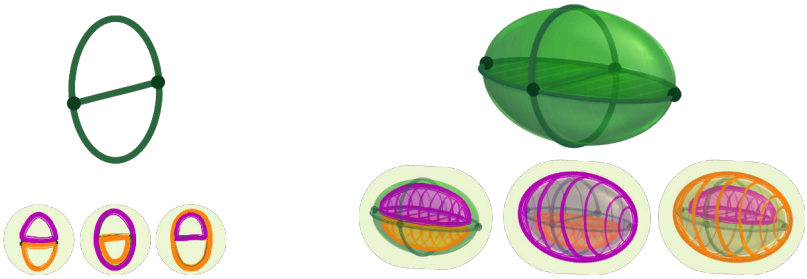


Figure 4.25 – Illustration of Conjecture 4.3.26: $\mathbf{Conf}_2(\mathrm{Susp}(K)) \approx \mathbf{Conf}_1(K)$

Conjecture 4.3.26. *The suspension operation lifts the configurations one dimension up:* $\forall q \geq 1, \mathbf{Conf}_{q+1}(\mathrm{Susp}(K)) \approx \mathbf{Conf}_q(K)$.

Conjecture 4.3.27. *The configurations of a wedge sum $K \vee K'$ (i.e. a one point union) is the Cartesian product of the configurations of K and K':* $\forall q, \mathbf{Conf}_q(K \vee K') \approx \mathbf{Conf}_q(K) \times \mathbf{Conf}_q(K')$.

More generally, gluing two complexes onto a p-cell (that is not a cycle) results in a Cartesian product of the q-configurations, with $q > q$.

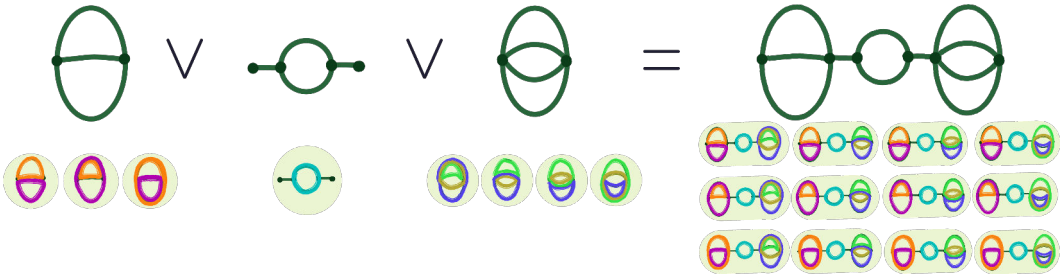


Figure 4.26 – Illustration of Conjecture 4.3.27

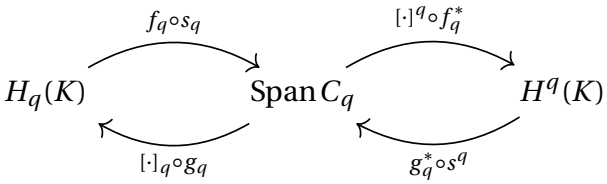
4.4. About the Constructiveness of Homology Morphisms

In this section, we formalize the concept of *isomorphisms constructible by HDVF* in the case of homology-cohomology duality and combinatorial duality. This section also establishes a bridge between Section 4.2 and Section 4.3.

4.4.1. Constructible homology-cohomology isomorphisms

It is a well-known property that, when they are finitely-generated, the q-homology group of a complex is isomorphic to the q-cohomology group (over a field or when no torsion arises). In our case, these two groups are $\mathbb{Z}/2\mathbb{Z}$ vector spaces of dimension β_q. Hence, the set of isomorphisms from $H_q(K)$ to $H^q(K)$ is in direct correspondence with the group $GL_{\beta_q}(\mathbb{Z}/2\mathbb{Z})$.

As mentioned above, a perfect HDVF for K provides such an isomorphism. The idea is that the homology group is in correspondence with the critical cells, which are in turn in correspondence with the cohomology group. This is summed up in the following commutative diagram:



Definition 4.4.1. Given a complex (K, ∂), a homology-cohomology isomorphism $\phi : H_q(K) \to H^q(K)$ is said to be *constructible by HDVF* (or simply *constructible*, for short) if there exists a perfect HDVF X, with reduction (f, g, h) such that $\phi = [\cdot]^q f^* f s_q$.

As every complex admits a perfect HDVF, the set of constructible homology-cohomology isomorphisms is nonempty. However, as illustrated in Figures 4.27 and 4.28, depending on the complex, some homology-cohomology isomorphisms are not constructible. This observation is mainly related to Section 2.3.4 and homology configurations.

In the following proposition, we show that these constructible isomorphisms are partly characterized by the homology configurations of K.

Proposition 4.4.1. *Let X and X' be two perfect HDVFs, with reductions (f,g,h) and (f',g',h'), for a complex K. Then,*

$$f'g \text{ is orthogonal} \iff [\cdot]^q f^* f s_q = [\cdot]^q f'^* f' s_q$$

Proof. Suppose $f'g$ is orthogonal, then $id = g^* f'^* \circ f'g$ so we have:

$$\begin{aligned}
[\cdot]^q f^* f s_q &= [\cdot]^q f^* g^* f'^* \circ f' g f s_q \\
&= [\cdot]^q (id - \partial h - h\partial)^* f'^* \circ f'(id - \partial h - h\partial) s_q && \text{by reduction properties} \\
&= [\cdot]^q (id - h^*\partial^*) f'^* f'(id - \partial h) s_q && \text{as } [\cdot]^q \partial^* = 0 \text{ and } \partial s_q = 0 \\
&= [\cdot]^q f'^* f' s_q && \text{as } \partial^* f'^* = 0 \text{ and } f'\partial = 0
\end{aligned}$$

Conversely, suppose that $[\cdot]^q f^* f s_q = [\cdot]^q f'^* f' s_q$, then we have:

$$\begin{aligned}
[\cdot]^q \circ f^* &= [\cdot]^q \circ f'^* \circ f' \circ s_q [\cdot]_q \circ g \\
[\cdot]^q \circ f^* &= [\cdot]^q \circ f'^* \circ f' g \\
id = g^* f'^* \circ f' g &\implies f'g \text{ orthogonal}
\end{aligned}$$

□

Thus, by Lemma 4.3.2 and since permutation matrices are orthogonal, two HDVFs belonging to the same configuration induce the same homology-cohomology isomorphisms. This immediately yields the following corollary:

Corollary 4.4.2. *There is a surjection from q-configurations to constructible q-homology-cohomology isomorphisms. Thus, for every complex K, the number of constructible q-homology-cohomology isomorphisms is bounded above by $|\mathbf{Conf}_q(K)|$.*

This corollary is illustrated in Figures 4.27 and 4.28, which use the following convention: the examples only include groups that are isomorphic to $\mathbb{Z}/2\mathbb{Z}^2$, for simplicity. Hence, these groups contain only four elements, three of them being non-zero. We represent each non-zero element with color red, blue or yellow. Isomorphisms are represented with arrows of the corresponding color. The colors of the source group are fixed, to help comparing different isomorphisms.

Remark. Every orthogonal matrix over $\mathbb{Z}/2\mathbb{Z}$ of dimension at most 3 is in fact a permutation matrix. Consequently, when $\beta_q \leq 3$, the condition that $f'g$ is orthogonal is equivalent to X and X' belonging to the same configuration. In this case, the constructible homology-cohomology isomorphisms are directly in bijection with the homology configurations. This is false in general, see Figure 4.29 for a counterexample when $\beta_q = 4$.

4.4.2. Constructible combinatorial duality isomorphisms

Here, we recover the setting of the combinatorial duality (Section 4.2.3): $(L,\partial) = (K^1,\partial^1) \oplus (K^0,\partial^0)$ with L acyclic, and K^1 closed in L under ∂. Let $X^0 = (P^0,S^0,C^0)$

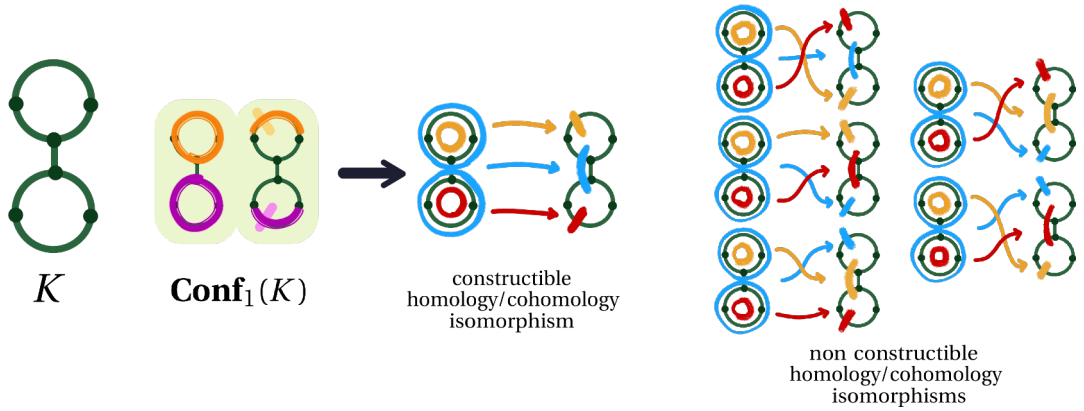


Figure 4.27 – Illustration of constructible and non constructible 1-homology-cohomology isomorphisms. This complex has only one 1-configuration (represented with both an explicit homology basis and a compatible explicit cohomology basis), so there is a unique constructible isomorphism $H_1(K) \to H^1(K)$. The five other isomorphisms $H_1(K) \to H^1(K)$ are not constructible, and are displayed on the right. Each isomorphism $H_1(K) \to H^1(K)$ is represented with colored arrows from non-zero homology classes to non-zero cohomology classes.

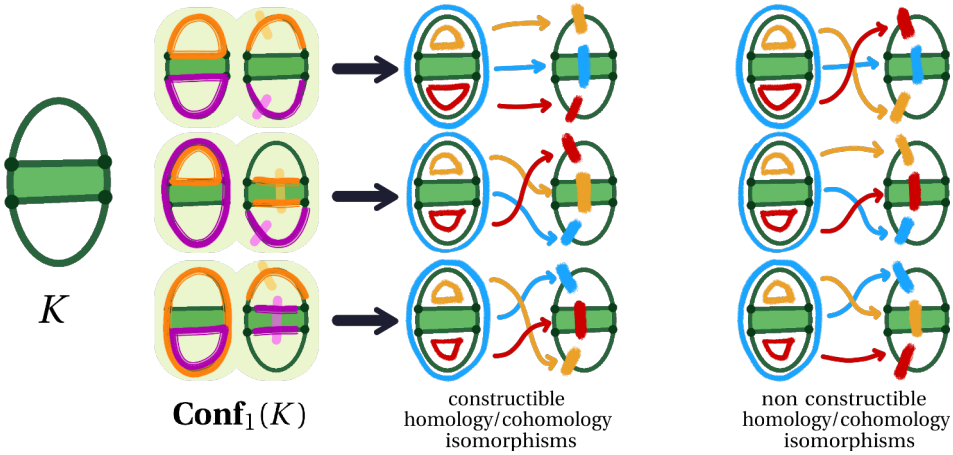


Figure 4.28 – This complex has three 1-configurations which produce three distinct constructible isomorphisms. The three remaining isomorphisms $H_1(K) \to H^1(K)$ are not constructible and displayed on the right.

and $X^1 = (P^1, S^1, C^1)$ be two perfect HDVFs for K^0 and K^1, respectively. By Theorem 4.2.6, $\alpha_{X^0}^{X^1}$ translates the isomorphisms $\operatorname{Span} C_{q+1}^0 \to \operatorname{Span} C_q^1$ into isomorphisms $H_{q+1}(K^0) \to H_q(K^1)$.

For clarity, an isomorphism D from $\operatorname{Span} C_{q+1}^0$ to $\operatorname{Span} C_q^1$ is called a *critical isomorphism*, whereas an isomorphism α from $H_{q+1}(K^0)$ to $H_q(K^1)$ is called a *combinatorial isomorphism*.

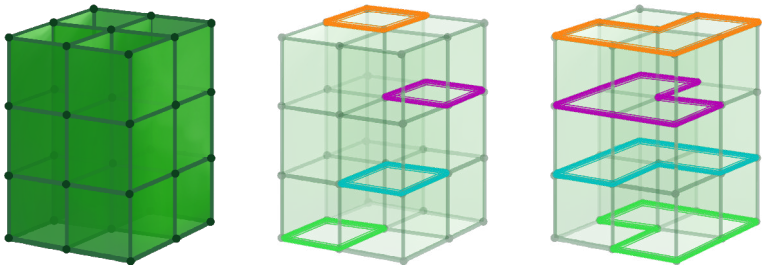


Figure 4.29 – A complex that admits less constructible isomorphisms $H_1(K) \to H^1(K)$ than its 1-configurations. Indeed, the middle and right images show two distinct 1-configurations, both represented with a representative explicit bases. However, the isomorphisms $H_1(K) \to H^1(K)$ induced by both this configuration are equal. Precisely, the reductions of two HDVFs, one in each configuration, induce the matrix $f'g = J_4 - id_4$, with J_4 being the 4×4 matrix filled with 1s. In $\mathbb{Z}/2\mathbb{Z}$, this matrix is orthogonal but is not a permutation. Thus, by Proposition 4.4.1, both HDVFs produce the same isomorphism $H_1(K) \to H^1(K)$, although they are not equivalent.

Defining the constructible critical and combinatorial isomorphisms

The standard critical isomorphism is $D^{X^1}_{X^0}$, and its corresponding combinatorial isomorphism is $\alpha^1_0 = \alpha^{X^1}_{X^0}(D^{X^1}_{X^0})$. $D^{X^1}_{X^0}$ can be obtained as the reduced boundary of the HDVF $X = X^0 \sqcup X^1$. It is possible to use this latter HDVF to build other critical isomorphisms. For instance, pairing C^0 cells to C^1 cells until X is perfect (i.e. a completion process) provides a bijection between C^0 and C^1, which in turn gives a combinatorial isomorphism. Note that this is how the homology quartets of Section 4.2.6 were computed in practice.

More generally, a completion X' of X provides a critical isomorphism. Indeed, suppose that $X = (P, S, C)$ and $X' = (P \sqcup \{\pi\}, S \sqcup \{\sigma\}, C \setminus \{\pi, \sigma\})$, with $\pi \in C_q$ and $\sigma \in C_{q+1}$, i.e. X' is obtained by performing a valid $\mathtt{A}^\sigma_\pi$ operation. Then, the reduced boundary d' of X' can be computed from the reduced boundary d of X. If we denote $\bar{\Pi} := C_q \setminus \{\pi\}$ and $\bar{\Sigma} := C_{q+1} \setminus \{\sigma\}$, we have:

$$d' = d^{\bar{\Pi}}_{\bar{\Sigma}} - d^{\bar{\Pi}}_{\sigma}(d^{\pi}_{\sigma})^{-1} d^{\pi}_{\bar{\Sigma}}$$

Note that d^π_σ is invertible because $\mathtt{A}^\sigma_\pi$ is a valid operation. This computation actually corresponds to a Gaussian elimination of the boundary matrix, as explained in GONZALEZ-LORENZO 2016(3.8). Therefore, performing a $\mathtt{A}^\sigma_\pi$ operation consists in performing the following matrix operation:

$$\begin{array}{ccc} \sigma & \bar{\Sigma} & \\ \begin{pmatrix} d^\pi_\sigma & d^\pi_{\bar{\Sigma}} \\ d^{\bar{\Pi}}_\sigma & d^{\bar{\Pi}}_{\bar{\Sigma}} \end{pmatrix} & \begin{matrix} \pi \\ \bar{\Pi} \end{matrix} \end{array} \longmapsto \begin{array}{cc} \sigma & \bar{\Sigma} \\ \begin{pmatrix} d^\pi_\sigma & 0 \\ 0 & d^{\bar{\Pi}}_{\bar{\Sigma}} - d^{\bar{\Pi}}_\sigma \left(d^\pi_\sigma\right)^{-1} d^\pi_{\bar{\Sigma}} \end{pmatrix} & \begin{matrix} \pi \\ \bar{\Pi} \end{matrix} \end{array}$$

From this observation, we can define the *constructible critical and combinatorial*

isomorphisms, see Figure 4.30 for an illustration. Firstly, we define inductively the *constructible critical isomorphisms*:

Definition 4.4.2. Given a pair (X^0, X^1) of perfect HDVFs for K^0 and K^1, we define the critical isomorphisms *constructible by* (X^0, X^1) by induction:

1. $D^{X^1}_{X^0}$ is constructible by (X^0, X^1).
2. If D is constructible by (X^0, X^1), and $D^\pi_\sigma \neq 0$ (with $\sigma \in C^1$ and $\pi \in C^0$), then the following isomorphism is constructible by (X^0, X^1):

$$\begin{array}{c} \sigma \quad\quad\quad\quad\quad\quad\quad\quad \bar{\Sigma} \\ \begin{pmatrix} D^\pi_\sigma & 0 \\ 0 & D^{\bar{\Pi}}_{\bar{\Sigma}} - D^{\bar{\Pi}}_\sigma \left(D^\pi_\sigma\right)^{-1} D^\pi_{\bar{\Sigma}} \end{pmatrix} \begin{array}{c} \pi \\ \bar{\Pi} \end{array} \end{array} \qquad \text{with } \bar{\Sigma} := C^1 \setminus \{\sigma\} \text{ and } \bar{\Pi} := C^0 \setminus \{\pi\}$$

This isomorphism can also be written as:

$$\tilde{D} = \mathtt{j}_\pi D \mathtt{i}_\sigma + \mathtt{j}_{\bar{\Pi}} \left(D - D \mathtt{i}_\sigma \left(D^\pi_\sigma\right)^{-1} \mathtt{j}_\pi D \right) \mathtt{i}_{\bar{\Sigma}}$$

Precisely, the constructible critical isomorphisms correspond to the pseudo-diagonalizations obtained by HDVF completion, starting from the HDVF $X = X^0 \sqcup X^1$.

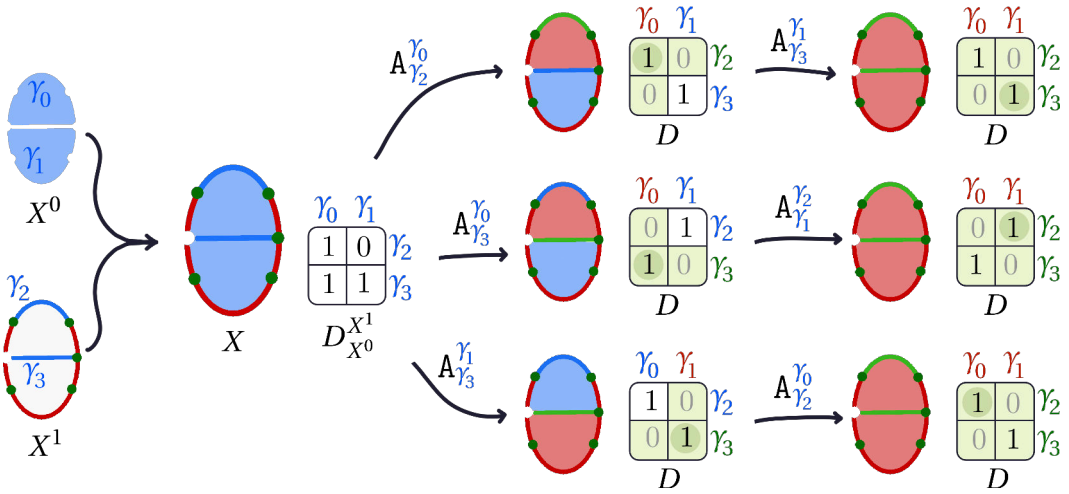


Figure 4.30 – An illustration of constructible critical isomorphisms induced by a pair (X^0, X^1) of perfect HDVFs. Alternatively, this illustrates the possible completion paths from X. Each $\mathtt{A}$ operation correspond to a Gaussian elimination. Note that different completion paths lead to the same critical isomorphisms. For instance, $\mathtt{A}^{\gamma_1}_{\gamma_3} X$ and $\mathtt{A}^{\gamma_0}_{\gamma_2} X$ produce the same D. In total, there are only three distinct critical isomorphisms constructible by (X^0, X^1), see Table 4.2 for their associated combinatorial isomorphisms.

Depending on the initial form of $D^{X^1}_{X^0}$, the number of critical isomorphisms that can be constructed may vary significantly. In some cases, a wide variety of critical

isomorphisms can be obtained, whereas in others only a few are possible. For instance, if $D^{X^1}_{X^0}$ is already a permutation matrix, then the second rule of Definition 4.4.2 produces the same isomorphism. In this situation, $D^{X^1}_{X^0}$ is the only critical isomorphism constructible by (X^0, X^1).

Considering constructible isomorphisms comes to a somewhat original question: what are the possible Gaussian elimination processes? Gaussian elimination steps are non necessarily commutative, hence, the combinatorics of constructible isomorphisms is much more complex than that of its individual steps. A simple heuristic for estimating the diversity of constructible isomorphisms by a fixed pair (X^0, X^1) is to count the number of entries equal to 1 in $D^{X^1}_{X^0}$. On the one hand, a large number of 1's suggests that there are many possible Gaussian eliminations, potentially yielding many constructible matrices D. On the other hand, when the number of 1's is small (close to the number of critical cells $|C^1_q|$), only a limited number of Gaussian eliminations are available, resulting in fewer constructible matrices D.

Definition 4.4.3. A combinatorial isomorphism $\alpha : H_{q+1}(K^0) \to H_q(K^1)$ is said to be *constructible by* (X^0, X^1) if there exists a critical isomorphism D constructible by (X^0, X^1) such that $\alpha = \alpha^{X^1}_{X^0}(D)$.

More generally, given K^0 and K^1, α is *constructible by HDVF* if there exists perfect HDVFs X^0 and X^1 such that α is constructible by (X^0, X^1).

First, α^1_0 is always constructible, by Proposition 4.2.2. Second, not all α are constructible, see for instance Table 4.3.

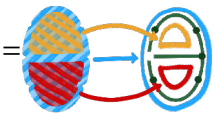
	Isomorphisms constructible by (X^0, X^1)		
Critical isomorphisms $D : \operatorname{Span} C^0_2 \to \operatorname{Span} C^1_1$	$D^{X^1}_{X^0} =$ $\begin{array}{cc} \gamma_0 & \gamma_1 \\ \hline 1 & 0 \\ 1 & 1 \end{array}$ $\begin{array}{l} \gamma_2 \\ \gamma_3 \end{array}$	$\begin{array}{cc} \gamma_0 & \gamma_1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}$ $\begin{array}{l} \gamma_2 \\ \gamma_3 \end{array}$	$\begin{array}{cc} \gamma_0 & \gamma_1 \\ \hline 0 & 1 \\ 1 & 0 \end{array}$ $\begin{array}{l} \gamma_2 \\ \gamma_3 \end{array}$
Combinatorial isomorphisms $\alpha^{X^1}_{X^0}(D) : H_2(K^0) \to H_1(K^1)$	$\alpha^1_0 =$ 		

Table 4.2 – Critical and combinatorial isomorphisms that are constructible by (X^0, X^1) (see Figure 4.30). As $H_2(K^0)$ and $H_1(K^1)$ are all isomorphic to $\mathbb{Z}/2\mathbb{Z}^2$, we use the same illustration convention as for Figures 4.27 and 4.28. Each γ_i is colored according to its corresponding homology class $[g(\gamma_i)]$ below.

Characterizing constructible critical and combinatorial isomorphisms with homology configurations

In what follows, we investigate the constructible isomorphisms and show that homology configurations partly characterize them. We still consider X^1 a perfect HDVF for K^1, but we now consider two perfect HDVFs X and X' of K^0. Denote (f^1, g^1, h^1), (f, g, h) and (f', g', h') the reductions of X^1, X and X'.

	Isomorphisms **not** constructible by (X^0, X^1)		
Critical isomorphisms $D : \operatorname{Span} C_2^0 \to \operatorname{Span} C_1^1$	$\begin{array}{cc\|c} \gamma_0 & \gamma_1 & \\ \hline 0 & 1 & \gamma_2 \\ 1 & 1 & \gamma_3 \end{array}$	$\begin{array}{cc\|c} \gamma_0 & \gamma_1 & \\ \hline 1 & 1 & \gamma_2 \\ 1 & 0 & \gamma_3 \end{array}$	$\begin{array}{cc\|c} \gamma_0 & \gamma_1 & \\ \hline 1 & 1 & \gamma_2 \\ 0 & 1 & \gamma_3 \end{array}$
Combinatorial isomorphisms $\alpha_{X^0}^{X^1}(D) : H_2(K^0) \to H_1(K^1)$			

Table 4.3 – Critical and combinatorial isomorphisms that are not constructible by (X^0, X^1). Indeed, no completion paths of $X = X^0 \sqcup X^1$ (see Figure 4.30) can produce the critical isomorphisms displayed in this table.

Denote $\phi_{X'}^X = f_q \circ g'_q : \operatorname{Span} C' \to \operatorname{Span} C$, which is an invertible matrix (with inverse $f'_q \circ g$). We have the following lemma:

Lemma 4.4.3. *For all critical isomorphisms $D : \operatorname{Span} C \to \operatorname{Span} C^1$ we have*

$$\alpha_X^{X^1}(D) = \alpha_{X'}^{X^1}\left(D\phi_{X'}^X\right)$$

Proof.

$$\begin{aligned}
\alpha_{X'}^{X^1}\left(D\phi_{X'}^X\right) &= [\cdot]^1 g^1 D f g' f' s^0 \\
&= [\cdot]^1 g^1 D f (id - \partial^0 h' - h' \partial^0) s^0 \\
&= [\cdot]^1 g^1 D f s^0 - [\cdot]^1 g^1 D f \partial^0 h' s^0 && \text{as } \partial^0 s^0 = 0 \\
&= [\cdot]^1 g^1 D f s^0 = \alpha_X^{X^1}(D) && \text{as } f\partial^0 = 0 \quad \square
\end{aligned}$$

When X and X' are in the same configuration, $\phi_{X'}^X$ is a permutation matrix, by Lemma 4.3.2. This yields the following proposition:

Proposition 4.4.4. *Suppose that $X \sim_q X'$, then we have:*

$$D \text{ is constructible by } (X, X^1) \implies D\phi_{X'}^X \text{ is constructible by } (X', X^1).$$

Proof. We prove this result by induction on D.
1. Suppose $D = D_X^{X^1}$. Then, by Proposition 4.2.2 and Lemma 4.4.3 we have

$$\alpha_{X'}^{X^1}\left(D_{X'}^X\right) = \alpha_0^1 = \alpha_X^{X^1}\left(D_{X'}^X\right) = \alpha_{X'}^{X^1}\left(D\phi_{X'}^X\right)$$

Hence, as $\alpha_{X'}^{X^1}$ is an isomorphism we have that $D_{X'}^X = D\phi_{X'}^X$, so it is constructible by (X', X^1).

2. Suppose that the following $\tilde{D}$ is constructible by (X, X^1):

$$\tilde{D} = \mathtt{j}_\pi D \mathtt{i}_\sigma + \mathtt{j}_{\bar{\Pi}} \left(D - D\mathtt{i}_\sigma \left(D_\sigma^\pi\right)^{-1} \mathtt{j}_\pi D\right) \mathtt{i}_{\bar{\Sigma}}$$

Where D is constructible by (X, X^1), $D^\pi_\sigma \neq 0$, $\sigma \in C$ and $\pi \in C^1$. We prove that $\tilde{D}\phi^X_{X'}$ is constructible by (X', X^1). By induction, $D\phi^X_{X'}$ is constructible by (X', X^1).

For simplicity, in what follows we write ϕ instead of $\phi^X_{X'}$. By Lemma 4.3.2, ϕ is a permutation, so define $\sigma' = \phi(\sigma) \in C'$. Thus, we have the commutativity properties: $\mathtt{i}_\sigma\phi = \phi\mathtt{i}_{\sigma'}$ and $\mathtt{i}_{\bar{\Sigma}}\phi = \phi\mathtt{i}_{\bar{\Sigma}'}$ (with $\bar{\Sigma} := C\backslash\{\sigma\}$ and $\bar{\Sigma}' := C'\backslash\{\sigma'\}$), which imply the following equalities:

$$\begin{aligned}
\tilde{D}\phi &= \mathtt{j}_\pi D\mathtt{i}_\sigma\phi + \mathtt{j}_{\bar{\Pi}}\left(D - D\mathtt{i}_\sigma\left(D^\pi_\sigma\right)^{-1}\mathtt{j}_\pi D\right)\mathtt{i}_{\bar{\Sigma}}\phi \\
&= \mathtt{j}_\pi D\phi\mathtt{i}_{\sigma'} + \mathtt{j}_{\bar{\Pi}}\left(D - D\mathtt{i}_\sigma\left(D^\pi_\sigma\right)^{-1}\mathtt{j}_\pi D\right)\phi\mathtt{i}_{\bar{\Sigma}'} \\
&= (D\phi)^\pi_{\sigma'} + \mathtt{j}_{\bar{\Pi}}\left(D\phi - D(\phi\mathtt{i}_{\sigma'}\phi^{-1})\left(\mathtt{j}_\pi D\mathtt{i}_\sigma\right)^{-1}\mathtt{j}_\pi D\phi\right)\mathtt{i}_{\bar{\Sigma}'} \\
&= (D\phi)^\pi_{\sigma'} + \mathtt{j}_{\bar{\Pi}}\left(D\phi - D\phi\mathtt{i}_{\sigma'}\left(\mathtt{j}_\pi D\mathtt{i}_\sigma\phi\right)^{-1}\mathtt{j}_\pi D\phi\right)\mathtt{i}_{\bar{\Sigma}'} \\
&= (D\phi)^\pi_{\sigma'} + \mathtt{j}_{\bar{\Pi}}\left(D\phi - D\phi\mathtt{i}_{\sigma'}\left(\mathtt{j}_\pi D\phi\mathtt{i}_{\sigma'}\right)^{-1}\mathtt{j}_\pi D\phi\right)\mathtt{i}_{\bar{\Sigma}'}
\end{aligned}$$

$D\phi$ is constructible by (X', X^1) (by induction), $(D\phi)^\pi_{\sigma'} \neq 0$ (because $(D\phi)^\pi_{\sigma'} = D^\pi_\sigma\phi$, $D^\pi_\sigma \neq 0$ and ϕ is invertible), $\sigma' \in C'$ and $\pi \in C^1$. Hence, we are in the second case of Definition 4.4.2, so $\tilde{D}\phi$ is constructible by (X', X^1). □

Lemma 4.4.3 and Proposition 4.4.4 could also be proved, by duality, with two HDVFs X and X' for K^1. We would have $\alpha^X_{X^0}(D) = \alpha^{X'}_{X^0}\left(\phi^{X'}_X D\right)$, and if $X \sim_{q+1} X'$, then

$$D \text{ is constructible by } (X^0, X) \quad \Longrightarrow \quad \phi^{X'}_X D \text{ is constructible by } (X^0, X').$$

Combining all these results with Definition 4.4.3, we obtain the following proposition:

Proposition 4.4.5. *Suppose that $X^0 \sim_{q+1} Y^0$ are perfect HDVFs for K^0, and $X^1 \sim_q Y^1$ are perfect HDVFs for K^1. Then, a combinatorial isomorphism α is constructible by (X^0, X^1) if and only if it is constructible by (Y^0, Y^1).*

Alternatively, homology configurations preserve the constructible combinatorial isomorphisms.

Homology configurations therefore provide a useful framework for understanding the homology isomorphisms that can be obtained between an object and its complement, using HDVF completion processes. For instance, all constructible combinatorial isomorphisms can be obtained by considering a single representative HDVF for each configuration.

This observation directly applies to the case of Alexander duality, as well as to the computation of homology quartets. The homology quartets produced by the method described in Section 4.2.6 correspond to triplets (X^0, X^1, Ψ) such that Ψ is a permutation matrix constructible by (X^0, X^1).

Proposition 4.4.5 partially answers the question raised in Section 4.4 concerning constructible isomorphisms, although several open problems remain.

Question 9. *Can we define a meaningful notion of* coherent *homology quartet? If so, how would this notion relate to that of* constructible quartets*?*

4.5. Conclusion

Section 4.2 presents Alexander duality through a constructive approach based on HDVFs. First, HDVFs are used to explicitly construct a morphism between the homology groups of a complex and those of its complement within a larger acyclic complex. This result, referred to as *combinatorial duality*, is then used to prove a constructive version of Alexander duality for geometric complexes. These approaches, based on HDVFs, are constructive in the sense that they explicitly provide isomorphisms underlying the considered duality. The latter duality is then used to derive a singular homology version of Alexander duality, for simplicial complexes.

More generally, this illustrates that HDVFs provide a combinatorial framework for explicitly constructing the morphisms involved in classical homological results. A simpler example is the homology-cohomology duality. Again, this highlights how HDVFs create a bridge between the homological and the combinatorial settings.

These constructions naturally raise several questions, in particular: which Alexander isomorphisms can be constructed using HDVFs? Can such isomorphisms be characterized?

Homology configurations, introduced in the following section, were initially studied in an attempt to address these questions. However, the properties of homology configurations slowly detached them from this original purpose, and established them as an object of study in their own right. From their definition, homology configurations allow us to classify HDVFs, explicit bases, and tri-partitions. This classification provides a higher point of view on the possible homology computations on a given complex. It has notably been shown that homology configurations coincide with cohomology configurations, which echoes with the result established in Section 2.3.4, and highlights the intrication between homology and cohomology computations.

A significant result is that homology configurations in maximal dimension are preserved under subdivisions of the complex. Unfortunately, this property does not hold in general for homology configurations in intermediate dimensions. Combined with the Hauptvermutung, this result implies that homology configurations in maximal dimension is a topological invariant for complexes of dimension at most 3.

Although configurations do not come with a natural algebraic structure, configurations in maximal dimension can be equipped with a graph structure. This structure allow them to be computed more efficiently than by performing a depth-first search on the graph of perfect HDVFs.

In conclusion, the study of HDVFs allows us to recover classical homological results in a constructive way, and also provides insights about the diversity of possible homology and cohomology computations. This fact also suggests several directions for future research. For example, what are the precise relationships between the isomorphisms constructible via HDVFs and the homology configurations?

Several open questions also remain concerning homology configurations. Do maximal-dimensional homology configurations still give topological information for complexes of dimension 4 or higher? Can we extract topological information from homology configurations of intermediate dimensions? Can homology configurations be used to study knots and links? For a more applied prospect, a better understanding of homology configurations could potentially provide new insights about methods that comprise several homology computations, such as multi-persistence or vines and vineyards.

Homology configurations also open the broader question of *combinatorial invariants*, i.e. structures that are preserved by subdivision of a complex. While this combinatorial approach to topology was common in the early of the 20th century, hoping to find interesting topological invariant, it might be relevant to study this topic from a computational perspective, with potential links with graph theory topics.

5. Geometric Measures for Homological Holes

Sommaire

5.1. Introduction . . . 179
5.2. Hole-Balls . . . 181
5.2.1. The signed-distance filtration . . . 181
5.2.2. Thickness, breadth and TB-pairs . . . 182
5.3. The Medial Axes . . . 183
5.3.1. Definition . . . 184
5.3.2. Geometrical and topological properties . . . 184
5.4. Computing Hole-Balls from the Medial Axes . . . 188
5.4.1. Overview of our approaches . . . 188
5.4.2. Link approach . . . 189
5.4.3. Medial axes approach . . . 190
5.4.3.1. Obtaining thickness balls from the inner medial axis . . . 191
5.4.3.2. Obtaining breadth balls from the outer medial axis . . . 194
5.5. Implementation and Validation on 3D Objects . . . 196
5.5.1. Discretization of our approaches . . . 196
5.5.2. Implementation and results . . . 198
5.5.3. Non-present hole-balls also gives information . . . 200
5.5.4. Limitation: obtaining an appropriate sampling . . . 200
5.5.5. Catching the right ball centers . . . 202
5.5.6. Complexity . . . 203
5.6. Prospects: Defining and Computing Hole Shapes . . . 204
5.7. Conclusion . . . 207

5.1. Introduction

In computational geometry, many works aim at analyzing, understanding or classifying geometrical shapes. To carry out this task, they need shape descriptors, which correspond to global or local information of the shape. Topology provides several theoretical descriptors, a large amount of which are highly relevant in the field of computational geometry. Among them, because of its computational properties, homology plays a major role. However, by definition, topology does not take into account geometry. That is why, in order to compute relevant descriptors, a collection of works

in computational topology aim at associating geometrical features to topological holes, see for instance T. K. DEY, LI, et al. 2008; T. K. DEY, HOU, et al. 2019; SONG et al. 2025; CHAMBERS et al. 2015. Such topological/geometric descriptors are useful for classification, as highlighted in CHAZAL, COHEN-STEINER, GUIBAS, et al. 2009.

The goal of this chapter is to extend previous works (see GONZALEZ-LORENZO, BAC, MARI, et al. 2016) which introduced two geometrical measures for topological holes in cubical complexes: namely the *thickness* and the *breadth* of a hole.

Intuitively, *thickness* represents the "fragility" of a hole and where to break it. *Breadth* represents the "size" of a hole and where to fill it. Thickness gives rise to a ball inside the object, while breadth gives a ball outside the object.

In the initial paper, those measures where defined and computed in the context of cubical complexes. However cubical complexes are topological structures that inherently integrate a notion of distance and geometry.

The first aim of this chapter is to extend the definition for non cubical objects. In addition, we propose ideas about how to geometrically measure topological holes, both theoretically and computationally, on volumetric objects.

For this purpose, we largely use the notion of *medial axis*. The *medial axis* is a well studied object that possesses both geometrical and topological properties. We investigate the idea that the computation of the medial axis, and its associated distance map, gives enough information to compute hole measures. Precisely, instead of performing persistent homology algorithms on the analyzed object (which would require a precise and expensive tetrahedralization), performing these algorithms on the medial axis is sufficient to obtain the thickness and breadth of the holes of the object.

We develop two different theoretical approaches, and implement a discrete version of them for experiments. On the one hand, the *link approach* intends to compute every thickness and breadth measures of an object as well as their homological link. On the other hand, the *medial axes approach* is a theoretically founded method that computes thickness and breadth measures without their homological link. This work has partially been published in a special issue for the *Symposium on Solid and Physical Modeling* in the *Computer Aided Design* journal, see GAZULL, BAC, et al. 2023.

The structure of the chapter is as follows. In Section 5.2 we define the *hole-balls* in the continuous case. Section 5.3 introduces the *medial axis* and its properties. Both of our approaches are described in Section 5.4 where we detail the theoretical foundation of both methods. In Section 5.5 we present experimental results and algorithmic stakes of our approach. Finally, Section 5.6 presents ongoing research works, based on similar ideas, where we try to define and compute the "geometrical shape" of a homological hole.

5.2. Hole-Balls

Persistent homology can provide geometrical information if the filtration itself has a geometrical meaning. As stated in the introduction, early works were carried out by GONZALEZ-LORENZO, BAC, MARI, et al. 2016 to define and use such a geometric filtration. However, these works heavily relied on the structure of cubical complexes. In this section we define the *hole-balls* for general objects in the euclidean space, using persistent homology and the *signed distance function*. Precisely, we consider objects in the *Alexandroff extension* of $\mathbb{R}^n$.

Definition 5.2.1 (ALEXANDROFF 1924)**.** The *Alexandorff extension* of $\mathbb{R}^n$ is $\mathbb{R}^n_* := \mathbb{R}^n \sqcup \{p_\infty\}$. $\mathbb{R}^n_*$ can be considered as a topological space by defining the opens of $\mathbb{R}^n_*$ as the opens of $\mathbb{R}^n$ together with all the sets $(\mathbb{R}^n \setminus K) \cup \{p_\infty\}$, where K is a compact of $\mathbb{R}^n$.

More precisely, we are mapping $\mathbb{R}^n$ to $\mathbb{R}^n_*$ by identifying all the points that are located at infinity to a single point p_∞. Particularly, we can consider that we have $\mathbb{R}^n \subset \mathbb{R}^n_*$. This construction implies that $\mathbb{R}^n_*$ is homeomorphic to the n-sphere. This property is mainly used to obtain an appropriate extension of Alexander duality in Section 5.4.3.

5.2.1. The signed-distance filtration

We now introduce the *signed distance function* and its associated filtration.

Definition 5.2.2. Given a set X in $\mathbb{R}^n_*$, the *signed distance function* of X is the following function:

$$\begin{aligned} sdf_X : \quad \mathbb{R}^n_* &\to \mathbb{R} \cup \{\pm\infty\} \\ x \in X &\mapsto -d(x, \partial X) \\ x \in X^c &\mapsto \; d(x, \partial X) \end{aligned}$$

With d the euclidean distance in $\mathbb{R}^n_*$ (that matches the distance on $\mathbb{R}^n$, and $\forall x,\ d(\mathrm{p}_\infty, x) = \infty$).

Definition 5.2.3 (sdf-filtration)**.** We define $\mathscr{F}_t(X) := sdf_X^{-1}([-\infty, t[)$. $(\mathscr{F}_t(X))_{t\in\mathbb{R}}$ is a filtration. We refer to it as the *sdf-filtration*. In mathematical morphology terms, this filtration corresponds to erosion if $t < 0$ and dilation if $t > 0$.

The persistent homology of the sdf-filtration provides different holes, which can be classified in four non-disjoint categories:

- *Early-birth holes*, whose birth date is before 0.
- *Late-death holes*, whose death date is after 0.
- *Late-birth holes*, whose birth date is after 0.
- *early-death holes*, whose death date is before 0.

The holes we are interested in are those in the object at $t = 0$, i.e. those in the intersection of early-birth holes and late-death holes. We refer to them as the *present holes*, they correspond to points in the upper-left quadrant of the persistence diagram (see Figure 5.1). Other holes are referred as *non-present holes*, they also give some interesting information, as highlighted in Section 5.5.3.

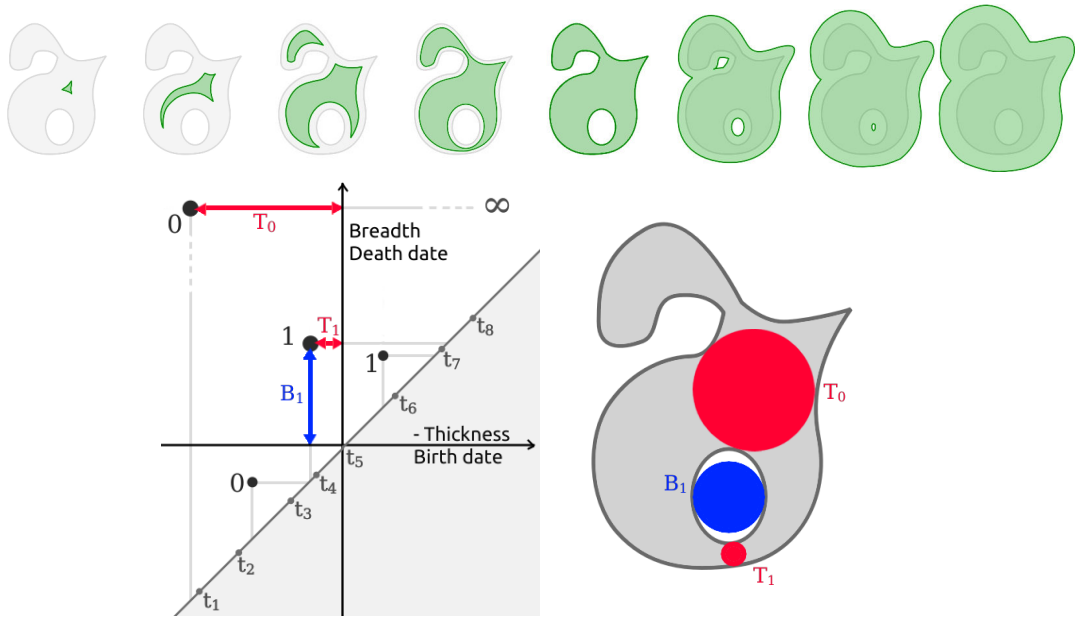


Figure 5.1 – The sdf-filtration (in green) of the object X for eight different values: $(t_1, \dots, t_8)$ with $t_5 = 0$. The persistent diagram is displayed on the bottom left: X has a 0-hole (born at before t_1 that does not die) and a 1-hole. Their respective balls are represented on the right (breadth in blue, thickness in red). The persistence diagram shows that two non-present holes appeared during the filtration, a 0-hole with lifetime within $[t_2, t_4]$ and a 1-hole with lifetime within $[t_5, t_7]$.

5.2.2. Thickness, breadth and TB-pairs

In this general setting, works of GONZALEZ-LORENZO, BAC, MARI, et al. 2016 actually appear as a special case (a discretization) of the sdf-filtration. Given a present hole with birth date x and death date y, we define the *thickness* $T := -x$ and the *breadth* $B := y$ of the hole.

Intuitively, the thickness of the hole corresponds to the fragility of the hole handle. The breadth, on the other side, corresponds to the size of the hole. Those measures can be associated with balls: the *T-ball* (respectively the *B-ball*) is the ball of radius T (respectively B) whose center is the point that induced the birth of the hole at time $-T$ (respectively the death of the hole at time $+B$). Such points are called *topologically critical points* and play a key role in our work. We refer to T-balls and B-balls as *hole-balls*. See Figure 5.1 for an illustration of this concepts.

Remark. In general, the topologically critical points are not unique (consider for instance a tubular annulus, with constant width). Though, SONG et al. 2025 proves that signed distance functions generically induce finitely many non degenerated critical points. In this case, the hole-balls are uniquely defined.

In the persistence diagram of the sdf-filtration, two different information are provided. On the one hand, the thickness and breadth values, on the other hand their homological relations which state which thickness corresponds to which breadth. We define the notion of *TB-pair* to integrate this subtle topological information. A TB-pair is a thickness value together with the breadth value corresponding to the same homological hole. Having only thickness and breadth values without every TB-pairs results in what we call a *partial persistence diagram* (see an illustration on Figure 5.2).

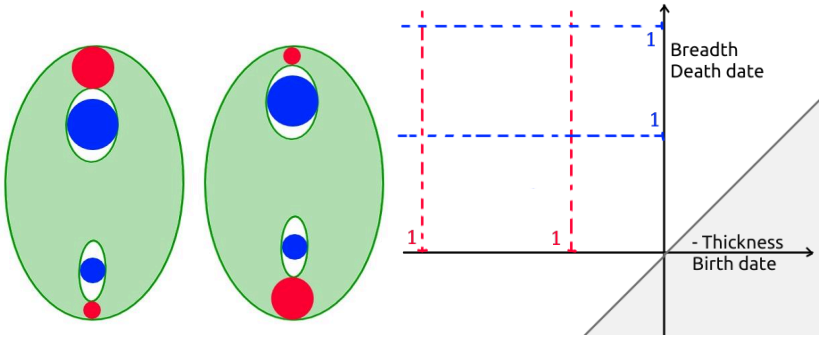


Figure 5.2 – Two objects (left) that have the same partial persistence diagram (right). They have different TB-pairs: in the first shape, the big B-ball is associated with the big T-ball whereas it is not the case in the second one.

The aim of this chapter is to study the properties of such hole measures and provide algorithms to compute them for nD shapes.

5.3. The Medial Axes

We now introduce the medial axis, a well known construction that carries both geometrical and topological information. The medial axis and its related constructions are used in practice in geometry processing for several purpose. One of its main application is shape reconstruction, see for instance PEETHAMBARAN et al. 2018; AMENTA and BERN 1998; T. K. DEY, GIESEN, et al. 2005; CHAZAL and LIEUTIER 2006.

First, we introduce the medial axes, and define an extension for $\mathbb{R}^n_*$. Second, we introduce the main topological property of the medial axis in Theorem 5.3.1. We then derive a result about the homology of the medial axes of objects that are sufficiently "regular", see Theorem 5.3.3. Finally, we investigate the literature about the stability and unstability properties of the medial axis. The computation of the medial axis is discussed later, in Section 5.5.

5.3.1. Definition

The medial axis of a set X is a geometric object that has a large number of characterizations, see for instance TAGLIASACCHI et al. 2016. We use the following definition:

Definition 5.3.1 (Set of closest boundary points)**.**

$$\begin{aligned} A_X : \mathbb{R}^n &\to \mathscr{P}(\partial X) \\ x &\mapsto \{s \in \partial X \mid d(x,s) = d(x,\partial X)\} \end{aligned}$$

Definition 5.3.2 (Medial axis)**.** The *medial axis* $\mathscr{M}(X)$ of X is the set of points in $\mathbb{R}^n$ that have two or more closest points on ∂X:

$$\mathscr{M}(X) = \left\{x \in \mathbb{R}^n \;/\; |A_X(x)| > 1\right\}$$

For this work, we consider a set $X \subset \mathbb{R}^n_*$ whose boundary ∂X is nonempty and bounded in $\mathbb{R}^n$. We thus require the following extension of the medial axis to $\mathbb{R}^n_*$.

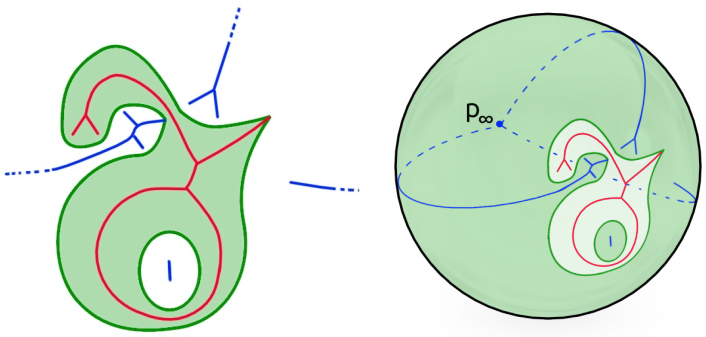


Figure 5.3 – An illustration of the extended medial axis $\mathscr{M}_*(X)$. In green: X (left) and X^c (right). In red: $\check{\mathscr{M}}_*(X)$. In blue: $\hat{\mathscr{M}}_*(X)$ which equals to $\check{\mathscr{M}}_*(X^c)$. $\mathbb{R}^2_*$ is mapped to the sphere S^2 for visual clarity.

Definition 5.3.3 (Extended medial axis)**.** The *extended medial axis* $\mathscr{M}_*(X)$ of X is defined as follows:

$$\mathscr{M}_*(X) = \mathscr{M}(X) \sqcup \{p_\infty\} \subset \mathbb{R}^n_*$$

We define the *inner medial axis* $\check{\mathscr{M}}_*(X)$ and the *outer medial axis* $\hat{\mathscr{M}}_*(X)$ (see Figure 5.3 for an illustration) as:

$$\begin{aligned} \check{\mathscr{M}}_*(X) &= \mathscr{M}_*(X) \cap X \\ \hat{\mathscr{M}}_*(X) &= \mathscr{M}_*(X) \cap X^c = \check{\mathscr{M}}_*(X^c) \end{aligned}$$

5.3.2. Geometrical and topological properties

Geometrically, the medial axis is known to be unstable with respect to the Hausdorff distance. Precisely, a small bump on the boundary of the shape can induce a spike

on the medial axis. Despite this limitation, stability results have been proved. In particular, CHAZAL and SOUFFLET 2004 proved that in 3D, $\mathscr{M}(X)$ is stable under C^2-perturbations when ∂X is C^2-smooth. Recently, DAL POZ KOUŘIMSKÁ et al. 2025 established the Lipschitz stability of $\mathscr{M}(X)$ under ambient diffeomorphisms.

While our theoretical work does not rely on the stability properties of the medial axis, it strongly relies on its topological properties. The main topological property of the medial axis is the following theorem, by LIEUTIER 2003. Recall that we write $A \equiv B$ iff A is homotopy equivalent to B, and $A \cong B$ iff A and B have isomorphic homology groups. In particular $A \equiv B \implies A \cong B$ (see in HATCHER 2002, p.111, Theorem 2.10).

Theorem 5.3.1 (LIEUTIER 2003)**.** *For all open X bounded in $\mathbb{R}^n$ we have $\check{\mathscr{M}}_*(X) \equiv X$.*

This result is particularly strong because it does not require any regularity assumptions on X. For our theoretical work, we require an extension of Theorem 5.3.1 to unbounded objects in $\mathbb{R}^n_*$ that have a bounded boundary in $\mathbb{R}^n$. Unfortunately, the proof of Theorem 5.3.1 can hardly be transposed for unbounded objects. For now, we achieve to extend homological properties of the medial axis to a specific class of objects that have some "regularity" and can be unbounded (see Theorem 5.3.3).

Let us first briefly introduce *subanalytic sets* and their subsequent properties. Subanalytic sets form one of the central examples of tame topology, see VAN DEN DRIES 1998 for an introduction to tame topology. We give here a brief definition of semianalytic and subanalytic sets, without details, please refer to BIERSTONE et al. 1988 for a proper definition.

Definition 5.3.4 (Semianalytic and subanalytic sets)**.** A *semianalytic* set is a subset of $\mathbb{R}^n$ define in terms of equalities and inequalities of analytic functions. A *subanalytic* set is a subset of $\mathbb{R}^n$ that is the image of a semianalytic set (of $\mathbb{R}^m$, with $m \geq n$) under a linear projection.

Proposition 5.3.2 (Properties of subanalytic sets)**.** *Let X and Y be subanalytic sets.*

1. *The closure $\overline{X}$, the interior $\mathring{X}$, the boundary ∂X, the complement $\mathbb{R}^n \setminus X$ and the medial axis $\mathscr{M}(X)$ are subanalytic (Theorem 4.1 and 2.1 in* CHAZAL *and* SOUFFLET *2004).*
2. *X is triangulable, its homology groups are finitely generated and it is locally contractible (see* HARDT *1976).*
3. *The union $X \cup Y$, the intersection $X \cap Y$ and the Cartesian product $X \times Y$ are subanalytic.*

We can now define a class of objects, encompassing a large variety of "real world" geometrical shapes, and whose regularity will support further results in this chapter.

Definition 5.3.5 (Regular object)**.** A set $X \subset \mathbb{R}^n_*$ is said to be a *regular object* if it satisfies the following properties:

1. X is a closed set of $\mathbb{R}^n_*$ whose boundary is nonempty bounded in $\mathbb{R}^n$.

2. X is a subanalytic set.
3. $\mathscr{M}_*(X)$ is closed in $\mathbb{R}^n_*$.

Regular objects offer wide modeling possibilities while avoiding topological "monsters" (due to 2. of Proposition 5.3.2). The third required property is the most restrictive assumption in practice, because it discards objects with sharp corners. Though, it allows smooth corners with arbitrarily large curvature.

We show the following theorem that states a homological link between an object and its medial axes:

Theorem 5.3.3 (Medial axes homology)**.** *Let X be a* regular object*, then*

$$\hat{\mathscr{M}}_*(X) \cong X^c \qquad \check{\mathscr{M}}_*(X) \cong \mathring{X}$$

First, we need Lemma 5.3.4. Given Y a nonempty closed subset of $\mathbb{R}^n_*$ whose boundary ∂Y is bounded in $\mathbb{R}^n$, we define the following function:

$$\begin{aligned} p_Y : \hat{\mathscr{M}}_*(Y)^c &\longrightarrow Y \\ y &\longmapsto \underset{x\in Y}{argmin}(d(x,y)) \end{aligned}$$

Lemma 5.3.4. *p_Y is well defined and continuous.*

Proof. p_Y is well defined. Indeed:

- if $y \in Y$: we simply have $p_Y(y) = y$.
- If $y \in \hat{\mathscr{M}}_*(Y)^c \setminus Y$: then $y \notin Y$ so we have $d(y,Y) = d(y,\partial Y)$ by closeness of Y. Moreover $y \in \mathscr{M}_*(Y)^c$ (because $\mathscr{M}_*(Y) = \hat{\mathscr{M}}_*(Y) \sqcup \check{\mathscr{M}}_*(Y)$ and $\check{\mathscr{M}}_*(Y) \subset Y$) so $A_Y(y)$ has only one element which is $p_Y(y)$.

Now we prove p_Y is continuous. Firstly, remind that distance functions are continuous. Take $y \in \hat{\mathscr{M}}_*(Y)^c$ and (y_n) a sequence of $\hat{\mathscr{M}}_*(Y)^c$ such that $y_n \overset{n\to\infty}{\longrightarrow} y$.
By continuity of distance we have :

$$d(y_n, Y) \overset{n\to\infty}{\longrightarrow} d(y, Y)$$

As we consider the sequence $(p_Y(y_n))_{n\in\mathbb{N}}$ in the compact space $\mathbb{R}^n_*$, Bolzano-Weierstrass theorem implies $Adh((p_Y(y_n))_{n\in\mathbb{N}}) \neq \emptyset$. Let's prove that $Adh((p_Y(y_n))_{n\in\mathbb{N}}) = \{p_Y(y)\}$:

Let $x \in Adh((p_Y(y_n))_{n\in\mathbb{N}})$. There exists some subsequence i_n such that $(y_{i_n}, p_Y(y_{i_n})) \overset{n\to\infty}{\longrightarrow} (y,x)$ so by continuity of distance:

$$d(y_{i_n}, p_Y(y_{i_n})) \overset{n\to\infty}{\longrightarrow} d(y,x)$$

As $d(y_{i_n}, Y) \overset{n\to\infty}{\longrightarrow} d(y,Y)$ and $d(y_{i_n}, Y) = d(y_{i_n}, p_Y(y_{i_n}))$, we have $d(y,x) = d(y,Y)$. Moreover $x \in Y$ (by closeness of Y).

Hence, by well-definition on p_Y, $x = p_Y(y)$ and $Adh((p_Y(y_n))_{n\in\mathbb{N}}) = \{p_Y(y)\}$. As a result, $p_Y(y_n) \overset{n\to\infty}{\longrightarrow} p_Y(y)$ therefore p_Y is continuous. □

Theorem 5.3.3 can now be proved, using a deformation retract based on p_X.

Proof. [Proof of Theorem 5.3.3] Let's X be a *regular object*. We can define the following deformation retract:

$$\begin{aligned} H : [0,1] \times \hat{\mathscr{M}}_*(X)^c &\longrightarrow \hat{\mathscr{M}}_*(X)^c \\ (t,y) &\longmapsto t p_X(y) + (1-t)y \end{aligned}$$

- H is well defined: Firstly, p_X is well defined. Indeed, by Definition 5.3.5-1, X is a nonempty closed subset whose boundary is bounded in $\mathbb{R}^n$.

  Secondly, for every t and y we have $H(t,y) \in \hat{\mathscr{M}}_*(X)^c$. To prove that, let y be an element of $\hat{\mathscr{M}}_*(X)^c$. Firstly, $\hat{\mathscr{M}}_*(X) \cap X = \emptyset$ so $p_X(y) \notin \hat{\mathscr{M}}_*(X)$. Let $t \in]0,1[$ and let $y_t := H(t,y)$. For every $x \in \mathbb{R}^n$ we denote $B(x)$ the minimal closed ball centered in x and intersecting ∂X. In particular, $B(x) \cap \partial X = A_X(x)$.

  If $y \in X$, then $y_t \notin \hat{\mathscr{M}}_*(X)$, so we consider the case where $y \notin X$. As y, y_t and $p_X(y)$ are aligned, we have that $B(y_t) \subset B(y)$, in particular, as $A_X(y) = \{p_X(y)\}$ and $A_X(y_t) \neq \emptyset$, we can deduced that $B(y_t) \cap \partial X = \{p_X(y)\}$. This implies that $y_t \notin \mathscr{M}_*(X)$.
- H verifies the properties of a deformation retract from $\hat{\mathscr{M}}_*(X)^c$ to X:
  - $X \subset \hat{\mathscr{M}}_*(X)^c$
  - $H(0,y) = y$
  - $H(1,y) = p_X(y) \in X$
  - $\forall t\ \forall y \in X,\ H(t,y) = y$
  - H is continuous (by Lemma 5.3.4).

As a consequence $\hat{\mathscr{M}}_*(X)^c \equiv X$. This directly implies:

$$\hat{\mathscr{M}}_*(X)^c \cong X \tag{5.1}$$

To conclude the proof, we want to use Alexander duality to "apply" the complementary operator on Equation (5.1). To do that, we use Corollary 5.4.3, an extension of Alexander duality that is introduced later.

X and $\hat{\mathscr{M}}_*(X)$ satisfies the assumptions for Corollary 5.4.3. Indeed, X is a closed nonempty subanalytic set of $\mathbb{R}^n_*$ by Definition 5.3.5. Moreover, $\hat{\mathscr{M}}_*(X)$ is closed and subanalytic as it is the intersection of two closed subanalytic sets $\mathscr{M}_*(X)$ and $\overline{X^c}$, by Proposition 5.3.2. Lastly, $\hat{\mathscr{M}}_*(X)$ is nonempty because it either contains p_∞ (if X is bounded) or a point in $\mathbb{R}^n$ (if X is unbounded).

Hence, using Corollary 5.4.3 on X and $\hat{\mathscr{M}}_*(X)$ gives for all q, $H_q(X^c) \approx H_{n-q-1}(X)$ and $H_q(\hat{\mathscr{M}}_*(X)^c) \approx H_{n-q-1}(\hat{\mathscr{M}}_*(X))$. Therefore, the homology isomorphisms

of Equation (5.1) translates to:

$$\hat{\mathscr{M}}_*(X) \cong X^c$$

$\overline{X^c}$ is also a *regular object*. Indeed, if p_∞ is in the interior of X, then $X^c = \mathbb{R}^n \setminus X$ is subanalytic (Proposition 5.3.2-1). If p_∞ is in the interior of X^c, then $X^c = \{p_\infty\} \cup \mathbb{R}^n \setminus X$ is still subanalytic. p_∞ can't be in ∂X because ∂X is bounded in $\mathbb{R}^n$. Therefore, using Proposition 5.3.2-1 again, $\overline{X^c}$ is subanalytic. Moreover, it is a regular object because $\partial X = \partial \overline{X^c}$ is nonempty bounded in $\mathbb{R}^n$.

All in all, applying the whole proof on $\overline{X^c}$ gives $\check{\mathscr{M}}_*(X) \cong \mathring{X}$. □

All in all, the medial axes characterize the homology of an object and its complement in the Alexandorff extension $\mathbb{R}^n_*$.

5.4. Computing Hole-Balls from the Medial Axes

5.4.1. Overview of our approaches

Computing the persistence diagram of the sdf-filtration and the topologically critical points associated to each hole is sufficient to obtain the hole-balls. We show that computing the persistence on a filtration built using the medial axis of our object is sufficient to obtain these information.

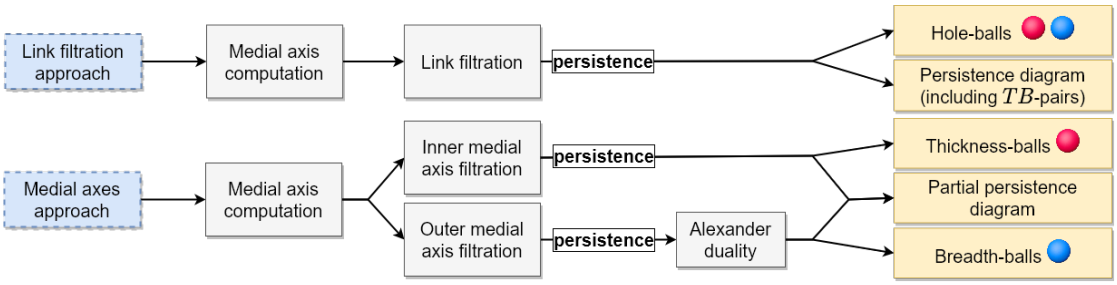


Figure 5.4 – Overview of the two approaches for measuring holes.

We develop two approaches, which are summarized in Figure 5.4.

1. The first one is the *link approach* and is developed in Section 5.4.2. It consists in building a single filtration that links the inner and outer medial axes along the sdf values. We conjecture that computing the persistence of this filtration gives every hole-ball and the full persistence diagram of the sdf-filtration. We can thus obtain the TB-pairs.
2. The second approach is the *medial axes approach* and is developed in Section 5.4.3. It consists in building two distinct filtrations: one based on the sdf-filtration restricted to the inner medial axis, the other based on the opposite sdf-filtration restricted to the outer medial axis.

We show that every T-ball can be obtained from the persistence of the inner medial axis filtration and every B-ball can be deduced from the persistence of the outer medial axis filtration using Alexander duality.

In practice, this approach is faster because the total size of the inner and outer medial axis filtration is smaller than the size of the filtration used in the link approach. Moreover, both persistence computations can be executed in parallel. In addition, this approach has a stronger theoretical foundations than the link approach. However, it only computes a partial persistence diagram and does not provide the TB-pairs. This might not be a problem, as the user might only be interested in the balls and not in their homological relations.

5.4.2. Link approach

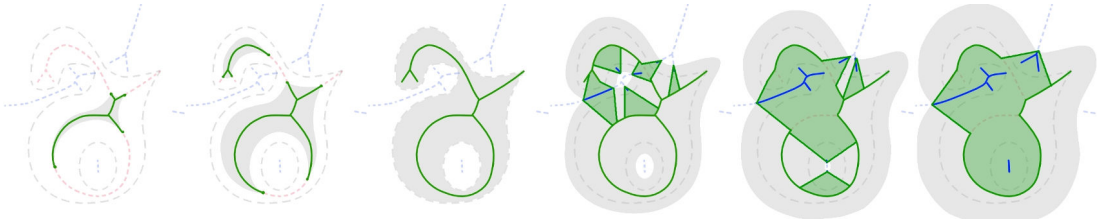


Figure 5.5 – The link filtration (in green) of an object X. The sdf-filtration is shown in grey and the outer medial axis in blue.

In the sdf-filtration, it appears that interesting homological information are localized on specific locations in space-time: topologically critical points and their associated birth and death dates. Therefore, the whole sdf-filtration is not necessary and we can focus on smaller sets. As the medial axis preserves homology and topologically critical points (see Section 5.4.3), it appears as a good candidate to build a simpler filtration.

The link approach was developed with the idea of reducing the sdf-filtration to its crucial part. It consists in a single filtration that links the inner medial axis to the outer one along the sdf values.

Definition 5.4.1. We define the following infinite bipartite graph $G_X(V,E)$ of X with:

$$V := \mathcal{M}_*(X) = \check{\mathcal{M}}_*(X) \sqcup \hat{\mathcal{M}}_*(X)$$
$$E := \left\{(\check{c},\hat{c}) \in \check{\mathcal{M}}_*(X) \times \hat{\mathcal{M}}_*(X) \;/\; A_X(\check{c}) \cap A_X(\hat{c}) \neq \emptyset\right\}$$

We define the link function:

$$\begin{array}{rccl} \xi : & \mathscr{P}(V) & \to & \mathscr{P}(\mathbb{R}^n_*) \\ & C & \mapsto & C \cup \displaystyle\bigcup_{(\check{c},\hat{c})\in E\cap C^2} \mathrm{Conv}(\check{c},\hat{c}) \end{array}$$

Definition 5.4.2 (Link filtration)**.** We define

$$\xi_t(X) := \xi\left(\mathscr{F}_t(X) \cap \mathscr{M}_*(X)\right)$$

$(\xi_t(X))_{t\in\mathbb{R}}$ is a filtration. We refer to it as the *link filtration* (see Figure 5.5).

We conjecture that the link filtration provides the same persistence diagram as the sdf-filtration and that it captures topologically critical points.

Conjecture 5.4.1 (Link filtration persistence)**.**

$$\mathscr{D}\left((\mathscr{F}_t(X))_{t\in\mathbb{R}}\right) = \mathscr{D}\left((\xi_t(X))_{t\in\mathbb{R}}\right)$$

In addition, the topologically critical points of $(\mathscr{F}_t(X))_{t\in\mathbb{R}}$ *are exactly the intersections of the topologically critical points of* $(\xi_t(X))_{t\in\mathbb{R}}$ *with the medial axes.*

The conjecture is proved for $t<0$, as the link filtration coincide with the *inner medial axis filtration* $(\mathscr{F}_t(X)\cap\check{\mathscr{M}}_*(X))_{t<0}$ introduced later in Section 5.4.3, see Theorem 5.4.8 and Proposition 5.4.9. Fortunately, experiments tend to validate the conjecture as pointed out in Section 5.5. The main advantage of this filtration is that it can be approached algorithmically using a Voronoi diagram, as explained in Section 5.5.

5.4.3. Medial axes approach

In spite of being essential for the computation of the TB-pairs, the link between both medial axes is not necessary to obtain hole-balls. Moreover, this link proves to be computationally expensive when dealing with the Voronoi filtration in practice. This motivates the *medial axes approach*, in which we discard the link and build two distinct filtrations: namely the *inner* and *outer medial axis filtrations* which respectively give T-balls and B-balls (but do not provide TB-pairs).

This method also relies on deeper theoretical foundations, as it is based on Alexander duality. Alexander duality has been investigated in Chapter 4. Let us recall here its standard formulation.

Proposition 5.4.2 (Alexander Duality)**.** *If X is a locally contractible nonempty compact of the n-sphere S^n, then:*

$$\forall q,\quad \tilde{H}_q(S^n\backslash X) \approx \tilde{H}^{n-q-1}(X)$$

Alexander duality is an important tool in this work. It justifies that studying objects in the *Alexandorff extension* of $\mathbb{R}^n$ is more relevant than in $\mathbb{R}^n$. Indeed, as $\mathbb{R}^n_*$ is homeomorphic to the n-sphere, Alexander duality can be considered in $\mathbb{R}^n_*$. Moreover, if an object has finitely generated homology groups, its q-homology and q-cohomology groups are isomorphic (see HATCHER 2002, Corollary 3.3). Therefore, using Proposition 5.3.2, we can deduce the following proposition:

Corollary 5.4.3 (Alexander Duality variant)**.** *If X is a nonempty closed subanalytic set of $\mathbb{R}^n_*$, then,*

$$\forall q, \quad \tilde{H}_q(X^c) \approx \tilde{H}_{n-q-1}(X)$$

5.4.3.1. Obtaining thickness balls from the inner medial axis

In this section, we deepen the link between the topology and the geometry of $\check{\mathcal{M}}_*(X)$ and X: the persistence diagram of the sdf-filtration on $]-\infty,0]$ is the same as the one obtained with the sdf-filtration restricted to $\check{\mathcal{M}}_*(X)$.
This result is stated in Theorem 5.4.8. It implies that computing the persistent homology of $\check{\mathcal{M}}_*(X)$ on $]-\infty,0]$ provides early-birth and early-death dates with their associated topologically critical points and therefore the T-balls of X.

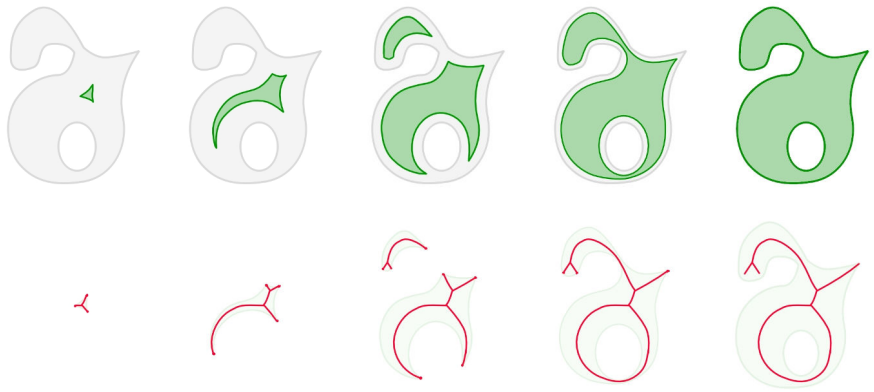


Figure 5.6 – The sdf filtration in $\mathbb{R}^2$: $(\mathscr{F}_t(X))_{t\in]-\infty,0]}$ (above). The filtration of the inner medial axis in $\mathbb{R}^2$: $\left(\check{\mathcal{M}}_t(X)\right)_{t\in]-\infty,0]}$ (below).

Definition 5.4.3 (Inner medial axis filtration)**.** Given t, we define $\check{\mathcal{M}}_t(X) := \mathscr{F}_t(X) \cap \check{\mathcal{M}}_*(X)$. The sequence $(\check{\mathcal{M}}_t(X))_{t\in]-\infty,0]}$ is a filtration. We refer to it as the *inner medial axis filtration*. Note that this definition is similar to the definition of the λ-medial axis Chazal and Lieutier 2005.

Proposition 5.4.4. *Given X a* regular object *and a filtration value $t<0$ such that $\mathscr{F}_t(X)$ is nonempty:*

$$\check{\mathcal{M}}_*(\mathscr{F}_t(X)) = \check{\mathcal{M}}_t(X)$$

Proving this result requires the two following lemmas. Let $t<0$ (case $t=0$ of Proposition 5.4.4 is trivial). For simplicity we write $\mathscr{F}_t$ instead of $\mathscr{F}_t(X)$.

Lemma 5.4.5.

$$\partial\mathscr{F}_t \subset sdf_X^{-1}(\{t\})$$

Proof. [Proof of Lemma 5.4.5] $\mathscr{F}_t$ is open because sdf is continue and p_∞ is in not in ∂X (∂X is bounded in $\mathbb{R}^n$). Moreover, the complementary of $sdf^{-1}([-\infty,t[)$ in $\mathbb{R}^n_*$ is $sdf^{-1}([t,+\infty])$.

By definition, the boundary of an open T is $\partial T = \overline{T} \cap T^c$, where $\overline{T}$ is the closure of T and T^c its complementary. Hence, the boundary of $\mathscr{F}_t$ is:

$$\partial \mathscr{F}_t = \overline{sdf^{-1}([-\infty, t[)} \cap sdf^{-1}([t, +\infty])$$

By continuity of sdf, the sdf value of a point in $\overline{sdf^{-1}([-\infty, t[)}$ is in $[-\infty, t]$. Therefore, the sdf value of a point in $\partial \mathscr{F}_t$ is in $[-\infty, t] \cap [t, +\infty] = \{t\}$. □

Lemma 5.4.6.

$$\forall c \in \mathscr{F}_t, \quad d(c, \partial X) = d(c, \partial \mathscr{F}_t) - t$$

Proof. [Proof of Lemma 5.4.6] Let us take c in $\mathscr{F}_t$.

- We choose $x \in A_{\mathscr{F}_t}(c)$ and $\hat{x} \in A_X(x)$. We have the following inequalities, by triangle inequality and using the definition of $A_{\mathscr{F}_t}(c)$ and $A_X(x)$:

$$\begin{aligned} d(c, \partial X) &\leq d(c, \hat{x}) \leq d(c, x) + d(x, \hat{x}) \\ &\leq d(c, \mathscr{F}_t) + d(x, \partial X) \end{aligned}$$

  $x \in \partial \mathscr{F}_t$ so $d(x, \partial X) = -t$ by Lemma 5.4.5. This implies:

$$d(c, \partial X) \leq d(c, \mathscr{F}_t) - t \tag{5.2}$$

- Conversely, by contradiction, suppose

$$d(c, \partial X) < d(c, \partial \mathscr{F}_t) - t \tag{5.3}$$

  Then, let $\hat{x} \in A_X(c)$. As $d(c, \partial X) > -t > 0$, we define:

$$\lambda := \frac{d(c, \partial X) - d(c, \partial \mathscr{F}_t) - t}{2d(c, \partial X)} \qquad x := \lambda c + (1 - \lambda)\hat{x}$$

  Thus, we obtain:

$$\begin{aligned} d(c, x) = (1 - \lambda) d(c, \hat{x}) &= \frac{d(c, \partial X) + d(c, \partial \mathscr{F}_t) + t}{2} \\ d(c, x) < d(c, \partial \mathscr{F}_t) &\quad \text{using Equation (5.3).} \end{aligned}$$

  So, as c is in $\mathscr{F}_t$, we have $x \in \mathscr{F}_t$. Moreover, we have:

$$\begin{aligned} d(x, \hat{x}) = \ \lambda \ d(c, \hat{x}) \ &= \frac{d(c, \partial X) - d(c, \partial \mathscr{F}_t) - t}{2} \\ d(x, \hat{x}) < -t &\quad \text{using Equation (5.3).} \end{aligned}$$

  Thus, $d(x, \partial X) \leq d(x, \hat{x}) < -t$, which implies $x \notin \mathscr{F}_t$. By contradiction, we obtain $d(c, \partial X) \geq d(c, \partial \mathscr{F}_t) - t$.

All in all, we obtain $d(c, \partial X) = d(c, \partial \mathscr{F}_t) - t$. □

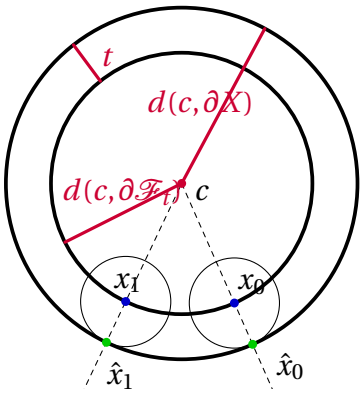


Figure 5.7 – Illustration of the different points defined in the proof of Proposition 5.4.4 and Lemma 5.4.6.

Proof. [Proof of Proposition 5.4.4]

- **Proof of** $\check{\mathscr{M}}_t(X) \subset \check{\mathscr{M}}(\mathscr{F}_t)$:
  let $c \in \check{\mathscr{M}}_t(X)$. We have directly $c \in \check{\mathscr{M}}(X)$ so we choose $\hat{x}_0, \hat{x}_1 \in A_X(c)$ such that $\hat{x}_0 \neq \hat{x}_1$. As $d(c,\partial X) > -t > 0$, we define:

$$x_0 = \mu c + (1-\mu)\hat{x}_0 \qquad x_1 = \mu c + (1-\mu)\hat{x}_1$$
$$\text{with} \quad \mu = \frac{-t}{d(c,\partial X)} \in]0,1[$$

  We easily have $x_0 \neq x_1$. To prove $c \in \check{\mathscr{M}}(\mathscr{F}_t)$, let us show that x_0 and x_1 belong to $A_{\mathscr{F}_t}(c)$. Let $i \in \{0,1\}$. We only need to show $d(x_i,c) = d(c,\partial\mathscr{F}_t)$ and $x_i \in \partial\mathscr{F}_t$:

$$\begin{aligned} d(x_i,c) = (1-\mu)d(c,\hat{x}_i) &= \frac{d(c,\partial X)+t}{d(c,\partial X)} d(c,\partial X) \\ &= d(c,\partial X) + t = d(c,\partial\mathscr{F}_t) \qquad \text{by Lemma 5.4.6} \end{aligned}$$

  Moreover, we have $d(x_i,\hat{x}_i) = \mu\, d(c,\hat{x}_i) = -t$. Hence, as $\hat{x}_i \in \partial X$, we obtain $d(x_i,\partial X) \leq d(x_i,\hat{x}_i) \leq -t$. This implies $x_i \notin \mathscr{F}_t$.

  x_i is not in $\mathscr{F}_t$ but $d(x_i,c) = d(c,\partial\mathscr{F}_t)$ (with $c \in \mathscr{F}_t$), therefore, x_i belongs to $\partial\mathscr{F}_t$.

- **Proof of** $\check{\mathscr{M}}(\mathscr{F}_t) \subset \check{\mathscr{M}}_t(X)$:
  let $c \in \check{\mathscr{M}}(\mathscr{F}_t)$. We choose $x_0, x_1 \in A_{\mathscr{F}_t}(c)$ such that $x_0 \neq x_1$. Let $i \in \{0,1\}$. Let $\hat{x}_i \in A_X(x_i)$. $x_i \in \partial\mathscr{F}_t$ so by Lemma 5.4.5 we have $d(\hat{x}_i,x_i) = -t$. By triangle inequality we have:

$$\begin{aligned} d(\hat{x}_i,c) &\leq d(\hat{x}_i,x_i) + d(x_i,c) \\ &\leq -t + d(c,\partial\mathscr{F}_t) \\ &\leq d(c,\partial X) \quad \text{by Lemma 5.4.6} \end{aligned} \tag{5.4}$$

  As $\hat{x}_i$ is in ∂X we obtain $d(c,\partial X) = d(\hat{x}_i,c)$ so $\hat{x}_i$ belongs to $A_X(c)$.

To conclude the proof, let us show that $\hat{x}_0 \neq \hat{x}_1$: the triangle inequality Equation (5.4) is in the equal case, which means that $\hat{x}_i$, x_i and c are co-linear. As $d(c,\partial X) > -t > 0$, we get:

$$x_0 = \mu c + (1-\mu)\hat{x}_0 \qquad x_1 = \mu c + (1-\mu)\hat{x}_1$$
$$\text{with} \quad \mu = \frac{-t}{d(c,\partial X)} \in]0,1[$$

Hence, we have $\hat{x}_0 \neq \hat{x}_1$ since $x_0 \neq x_1$. □

Proposition 5.4.4 implies the following property:

Corollary 5.4.7. *Given X a* regular object *and $t \leq 0$ such that $\mathscr{F}_t(X) \neq \emptyset$:*

$$\mathscr{F}_t(X) \cong \check{\mathscr{M}}_t(X)$$

Proof. $\overline{\mathscr{F}_t(X)}$ is a *regular object*. Indeed, it is closed with nonempty bounded boundary, subanalytic (see CHAZAL and LIEUTIER 2005) and its medial axis is closed (by Proposition 5.4.4). Theorem 5.3.3 on $\overline{\mathscr{F}_t(X)}$ gives:

$$\check{\mathscr{M}}_*(\mathscr{F}_t(X)) \cong \mathscr{F}_t(X)$$

Combined to Proposition 5.4.4 we get the desired result. □

As a consequence, the sdf-filtration persistence diagram of X on $]-\infty,0]$ can be obtained by performing the persistence algorithm on its inner medial axis.

Theorem 5.4.8 (Hole measures from inner medial axis)**.** *Given X a* regular object *of $\mathbb{R}^n_*$:*

$$\mathscr{D}\left((\mathscr{F}_t(X))_{t\in]-\infty,0]}\right) = \mathscr{D}\left(\left(\check{\mathscr{M}}_t(X)\right)_{t\in]-\infty,0]}\right)$$

The studied interval is $]-\infty,0]$, so this diagram gives every early-birth and early-death date, but we also need the ball centers in order to fully obtain the T-balls. Fortunately, we have the following proposition, which can be deduced from results in CAZALS et al. 2008; SIERSMA 1996 stating that topologically critical points lie on the medial axis.

Proposition 5.4.9. *The topologically critical points of $(\mathscr{F}_t(X))_{t\in]-\infty,0]}$ are exactly the topologically critical points of $\left(\check{\mathscr{M}}_t(X)\right)_{t\in]-\infty,0]}$.*

As a result, the persistence algorithm of $\left(\check{\mathscr{M}}_t(X)\right)_{t\in]-\infty,0]}$ fully provides the T-balls.

5.4.3.2. Obtaining breadth balls from the outer medial axis

In this part we aim at capturing the persistence on $[0,+\infty[$ (i.e. the B-balls, late-death and late-birth dates) using the outer medial axis and Alexander duality. To begin with, given our *regular object* X, we define the filtration of the complement:

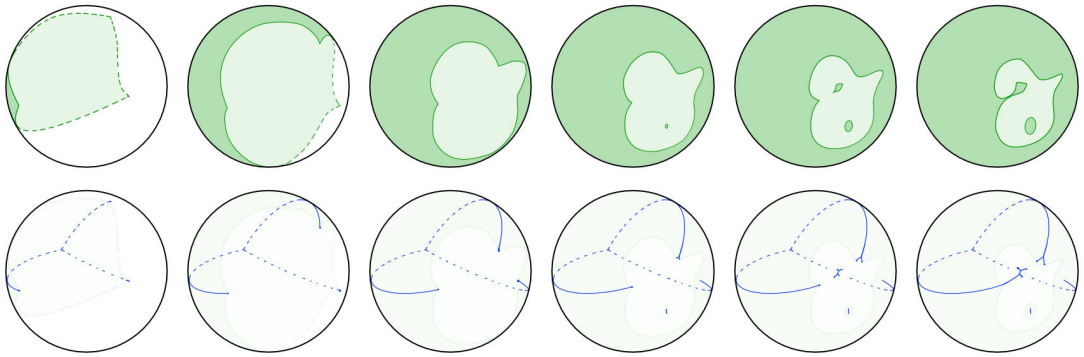


Figure 5.8 – The sdf^c filtration in $\mathbb{R}^2_*$: $(\mathscr{F}_t(X^c))_{t\in]-\infty,0]}$ (above). The filtration of the outer medial axis in $\mathbb{R}^2_*$: $\left(\check{\mathscr{M}}_t(X^c)\right)_{t\in]-\infty,0]}$ (below). $\mathbb{R}^2_*$ is displayed as the 2-sphere.

Definition 5.4.4 (sdf^c filtration)**.** $(\mathscr{F}_t(X^c))_{t\in\mathbb{R}}$ is a filtration. We refer to it as the *sdf^c filtration* (see Figure 5.8). Intuitively, erosions in the sdf-filtration correspond to dilations in the sdf^c filtration and vice-versa.

Theorem 5.4.8 states that the persistence diagram of the sdf-filtration of X on $]-\infty,0]$ can be obtained from the diagram of the sdf-filtration restricted to $\check{\mathscr{M}}_*(X)$. Conversely, we now prove Theorem 5.4.12, which states that the persistence diagram of the sdf-filtration of X on $[0,+\infty[$ can be deduced from the diagram of the sdf^c-filtration restricted to $\hat{\mathscr{M}}_*(X)$.

First, we need the two following lemma. Applying Corollary 5.4.7 to $\overline{X^c}$ gives the following lemma.

Lemma 5.4.10. *Given X a* regular object *and $t \leq 0$ such that $\mathscr{F}_t(X^c) \neq \emptyset$:*

$$\mathscr{F}_t(X^c) \cong \check{\mathscr{M}}_t(X^c)$$

As in Section 5.4.3.1, this implies that the persistence diagram of X^c on $]-\infty,0]$ can be obtained by computing the persistence diagram of the inner medial axis of X^c.

Then, we use an extension of Alexander duality to deduce the persistence diagram of X on $[0,+\infty[$. Formally, applying Theorem 3.2 of Kališnik 2013 to the filtration $(\mathscr{F}_t(X))_{t\in[0,+\infty[}$ gives the following lemma.

Lemma 5.4.11 (Persistent Alexander duality)**.**

$$\mathscr{D}\left((\mathscr{F}_t(X))_{t\in[0,+\infty[}\right) = \mathscr{D}^c\left(\left(\mathscr{F}_t(X^c)\right)_{t\in]-\infty,0]}\right)$$

Where $\mathscr{D}^c(\cdot)$ stands for the dual of the underlying persistence diagram following Alexander duality: the i-holes of coordinates (x,y) become $(n-i-1)$-holes of coordinates $(-y,-x)$ (with a special case for 0-dimensional holes).

Remark. $(\mathscr{F}_t(X))_{t\in[0,+\infty[}$ satisfies the necessary assumption of KALIŠNIK 2013 because every $\mathscr{F}_t(X)$ is subanalytic.

All in all, combining Lemmas 5.4.10 and 5.4.11, we obtain the following theorem.

Theorem 5.4.12 (Hole measures from outer medial axis)**.** *Given X a regular object of $\mathbb{R}^n_*$,*

$$\mathscr{D}\left((\mathscr{F}_t(X))_{t\in[0,+\infty[}\right) = \mathscr{D}^c\left(\left(\check{\mathscr{M}}_t(X^c)\right)_{t\in]-\infty,0]}\right)$$

In particular, computing the persistent homology on $\hat{\mathscr{M}}_*(X)$ with the sdf^c-filtration provides every late-death and late-birth dates of X.

To fully find the B-balls of X, we also need a proposition analogous to Proposition 5.4.9, which can be deduced from results in CAZALS et al. 2008; SIERSMA 1996 combined with Lemma 5.4.11:

Proposition 5.4.13. *The topologically critical points of $(\mathscr{F}_t(X))_{t\in[0,+\infty[}$ are exactly the topologically critical points of $\left(\check{\mathscr{M}}_t(X^c)\right)_{t\in]-\infty,0]}$, but they appear in the reverse order.*

Hence, the persistence diagram of the outer medial axis $\hat{\mathscr{M}}_*(X)$ with the sdf^c-filtration induces every late-death and late-birth date of X with their critical points. Therefore, it fully provides the B-balls.

All in all, computing the persistence of the inner and outer medial axes of X (using their appropriate filtration) gives enough information to obtain every hole-ball of X. Though, in general, this information is not sufficient to recover the whole persistence diagram.

5.5. Implementation and Validation on 3D Objects

5.5.1. Discretization of our approaches

The medial axis is central in our work. For implementation purpose, a discrete representation of the medial axis is needed. Since exact computation of the medial axis is computationally expensive (CULVER et al. 2004), we actually approximate it.

Computation of the medial axis

The computation of the medial axis is still an active research topic, and many algorithms have been proposed in the literature, as exposed for instance in TAGLIASACCHI et al. 2016.

A large number of them rely on the notion of *Voronoi diagram* (and its dual the *Delaunay triangulation*) of a sampling of the object's boundary. The main idea is that Voronoi vertices of a sufficiently dense sampling of ∂X tend to lie close to $\mathscr{M}(X)$, although exceptions can occur in $\mathbb{R}^n$, $n \geq 3$, see ATTALI et al. 2009. These methods typically filter the Voronoi diagram or select a subset of it. Such methods include AMENTA, CHOI, et al. 2000; CHAZAL and LIEUTIER 2005; T. K. DEY and ZHAO 2004, as well

as GIESEN, RAMOS, et al. 2011, which introduces the *core*, a related structure with properties well suited to our work.

These methods usually provide topological guarantees under assumptions on the initial sampling. Precisely, they require an ε-sampling, with sufficiently small ε, see Definition 5.5.2 and Fig. 5.9. We also mention the method of WANG et al. 2024, which computes an approximation of the medial axis from a power diagram, and then revises it to correct its topological mistakes.

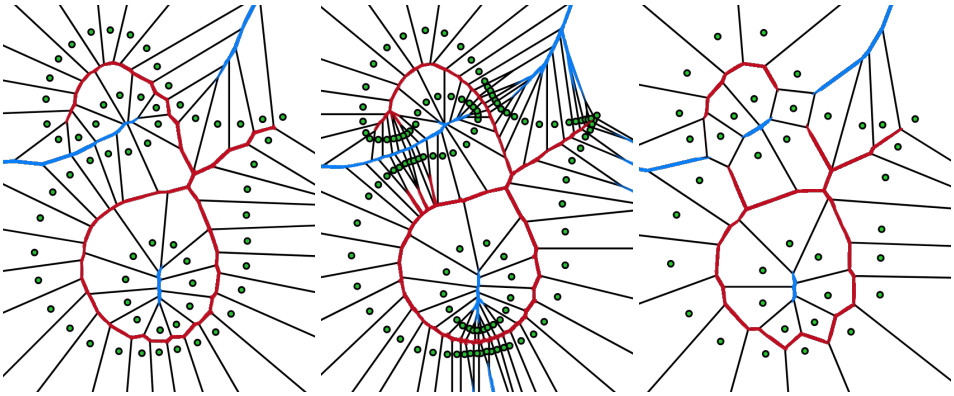


Figure 5.9 – Three samplings of the surface and their Voronoi diagrams, together with the resulting medial axes approximation in red (inner) and blue (outer). The first sampling is a uniformly dense sampling. The second one is an ε-sampling, whose density adapts to the distance to the medial axis. The third one is a sampling not dense enough to ensure topological correctness of the medial axes approximation.

Recently, learning-based approaches also emerged, such as KONG et al. 2024 and CLÉMOT and DIGNE 2023, which computes the medial axis from an implicit neural representation of the sdf. Other 3D medial axis computation methods include variational approaches (HUANG et al. 2024), voxel-based approaches (YAN et al. 2018) or optimization techniques (REBAIN et al. 2019).

The Voronoi filtration and the discrete medial axes filtrations

For our purposes, we require an approximation of the medial axes that both preserves the topology of the input object and provides a structure for homology computations. For this reason, we choose an approach based on the Voronoi diagram of a sampling of ∂X. Indeed, the Voronoi diagram induces a CW-complex structure on which we can build filtrations and perform persistent homology computations. Moreover, it provides topological guarantees when the sampling is sufficiently dense (see Section 5.5.4) and gives enough information to capture the relevant topologically critical points (see Section 5.5.5).

Regarding the link approach, the Voronoi diagram of a surface sampling directly induces an approximation of the link function. More precisely, the duals of Delaunay faces lying on ∂X give an approximation of ξ. We define the *Voronoi filtration*, which can be considered as a discrete version of the link filtration:

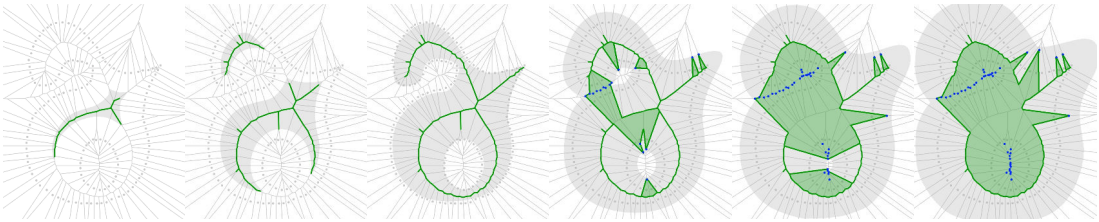


Figure 5.10 – The Voronoi filtration of a sampling of ∂X, which can be understood as the discrete counterpart of the link filtration in Figure 5.5. The blue points represent the Voronoi vertices that approximate the outer medial axis.

Definition 5.5.1 (Voronoi Filtration)**.** Consider the nD Voronoi diagram of a sampling on ∂X. We build a filtration by assigning a value to each Voronoi cell: the value of a Voronoi vertex is its sdf value. The value of a q-cell c is defined as the maximum of the values of its $(q-1)$-dimensional boundary cells. The resulting filtration is called the *Voronoi filtration* and is the discrete counterpart of the link filtration. See Figure 5.10 for an illustration.

Regarding the medial axes approach, we consider the medial axis approximation given by the Voronoi cells that do not intersect ∂X. For sake of simplicity, we do not apply any additional filtering to this approximation, unlike in CHAZAL and LIEUTIER 2005; AMENTA, CHOI, et al. 2000; GIESEN, RAMOS, et al. 2011, as we believe that such filtering does not affect the resulting hole-balls. The inner and outer medial axis filtrations are then defined using the sdf values assigned to Voronoi vertices.

> *Remark.* In practice, these discrete filtrations based on Voronoi diagrams may fail to converge to the desired TB-pairs, even when the sampling density tends to infinity. To address this issue, it is necessary to refine the filtration by incorporating not only Voronoi vertices but also *Delaunay critical points*, as discussed in Section 5.5.5.

5.5.2. Implementation and results

We implemented the two methods for 3D models in C++ using *CGAL* (THE CGAL PROJECT 2025) for geometry and *PHAT* BAUER et al. 2014 for persistent homology. We experimented our methods on different 3D models, mostly from the Thingi10k ZHOU et al. 2016 dataset. We focused on objects that we regarded as topologically interesting (non trivial shapes with multiple holes of different dimensions) and we used uniform samplings with different densities to build the medial axis approximation using Voronoi diagram. The code can be find on a GitHub repository GAZULL, GONZALEZ-LORENZO, et al. 2023.

Given a correct approximation of the medial axis, the results obtained using the medial axes approach are valid. We tested the method on tricky objects such as the *knot* Figure 5.14), which seems complicated but has only one 1-dimensional hole. We also tested the method on *egg-in-a-box* Figure 5.12), which consists in a component within another component. The algorithm returned the appropriate 0-dimensional

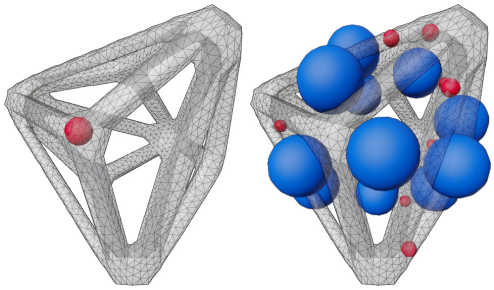


Figure 5.11 – 0-dimensional (left) and 1-dimensional (right) hole-balls of *triakis-tetrahedron*.

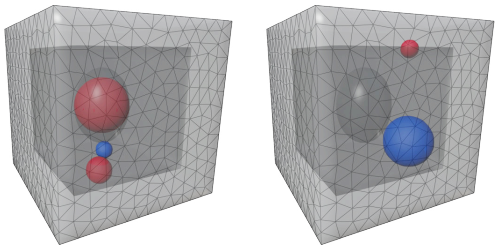


Figure 5.12 – 0-dimensional (left) and 2-dimensional (right) hole-balls of *egg-in-a-box*.

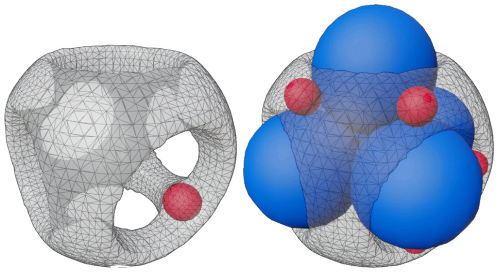


Figure 5.13 – 0-dimensional (left) and 1-dimensional (right) hole-balls of a *hollow-cubic-ball*.

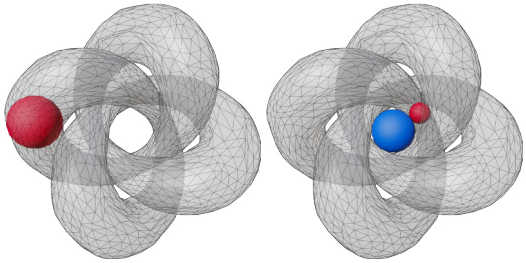


Figure 5.14 – 0-dimensional (left) and 1-dimensional (right) hole-balls of *knot*.

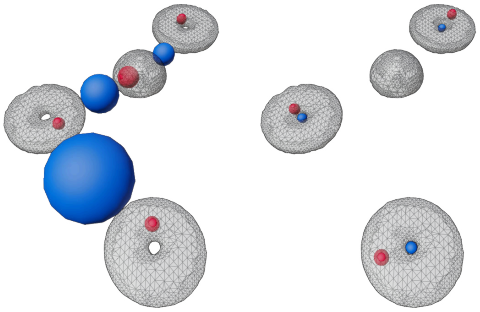


Figure 5.15 – 0-dimensional (left) and 1-dimensional (right) hole-balls of *four-components* with three 1-holes.

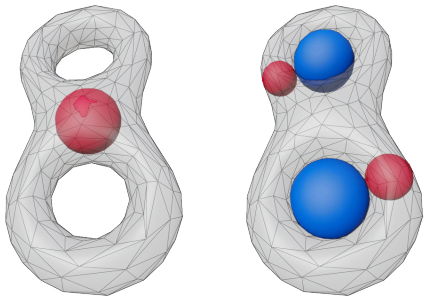


Figure 5.16 – 0-dimensional (left) and 1-dimensional (right) hole-balls of *eight*.

hole-balls corresponding to the two components and the 2-dimensional hole-balls corresponding to the inside of the box. We tested the approach on symmetrical objects such as the *hollow-cubic-ball* Figure 5.13) and the *triakis-tetrahedron* Figure 5.11). As expected, the algorithm returned 1-dimensional T-balls (respectively B-balls) with almost exactly the same radii.

Given the same approximation of the medial axis, the link filtration and the medial

axes approach gives similar results: nearly the same thickness and breadth values and most of the time close ball centers. In addition, the link filtration provided the expected TB-pairs. For instance, in the *four-components* object Figure 5.15), each 0-dimensional T-ball inside a component is related to the 0-dimensional B-ball that links this component to the others. Similarly, the 1-dimensional T-ball of a holy component is related to the 1-dimensional B-ball in the same component. In the *eight* object Figure 5.16), each 1-dimensional T-ball is related to its tangent B-ball.
All in all, this provides an experimental validation of the Link approach, which is not theoretically established (see Conjecture 5.4.1).

Notice that in some cases, the ball centers returned by the two methods are far apart. This is due to the unstability of topologically critical points which might not be unique, for instance on long portions with a constant distance to medial axes (as between the rings in Figure 5.17 Figure 5.17).

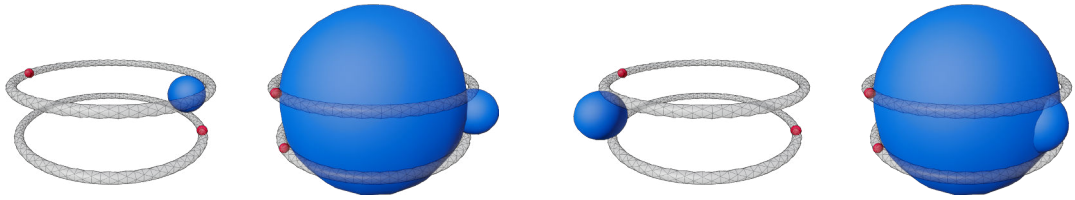


Figure 5.17 – Comparison between the hole-balls computed by both approaches. Two images on the left: 0-dimensional (left) and 1-dimensional (right) hole-balls of *two-rings* using **medial axes approach**. Two images on the right: 0-dimensional (left) and 1-dimensional (right) hole-balls of *two-rings* using **link approach**.

5.5.3. Non-present hole-balls also gives information

Both of our methods compute not only the actual holes of an object but also every late-birth and early-death holes. It is possible to extend the definition of hole-balls to these non-present holes, potentially inducing negative thickness or breadth.

Even though these holes are not present in the object, their hole-balls give interesting intuitive geometrical information. For instance, non-present holes of dimension 0 can give a hint about how to segment an object in a meaningful way (see Figure 5.18(a)). Non-present holes of dimension 2 can tell where the object looks like a “cage” and where is the easiest place for a ball to escape it (see Figure 5.18(b)). It may be interesting to investigate this idea a bit further, to maybe use our method for general topological feature detection.

5.5.4. Limitation: obtaining an appropriate sampling

The major practical limitation to our approach is the quality of the sampling. In order to obtain the theoretical homological guarantees that ensure the correctness of our methods, we need a sufficiently accurate sampling. Precisely, the perfectly suited

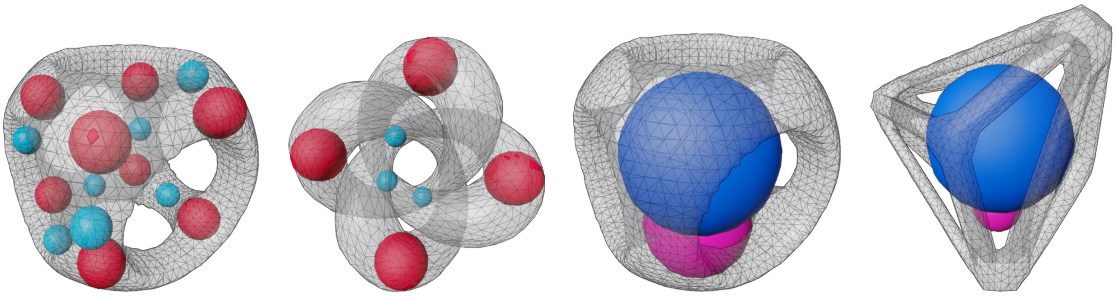


Figure 5.18 – Two images on the left: hole-balls of 0-holes (including non-present holes) for *hollow-cubic-ball* and *knot*. Light blue balls are B-balls with $B < 0$. Two images on the right: hole-balls of non-present 2-holes for *hollow-cubic-ball* and *triakis-tetrahedron*. Magenta balls are T-balls with $T < 0$.

sampling would be an *ε-sampling*, that is, a sampling whose density varies according to the distance to the medial axis (see Figure 5.9):

Definition 5.5.2 (GIESEN, RAMOS, et al. 2011)**.** An ε-sampling of the ∂X is a set $P \subset \partial X$ such that:

$$\forall x \in \partial X,\ \exists p \in P\ /\quad d(x,p) \leq \varepsilon \cdot d(x, \mathscr{M}_*(X))$$

It has been showed in GIESEN, RAMOS, et al. 2011; AMENTA, CHOI, et al. 2000 that an ε-sampling with ε small enough provides topological guarantees. In practice, the choice of density is important: a sampling not sufficiently dense would result in wrong hole-balls (see Figure 5.19 and Figure 5.19), whereas an overly dense one could be too computationally expensive.

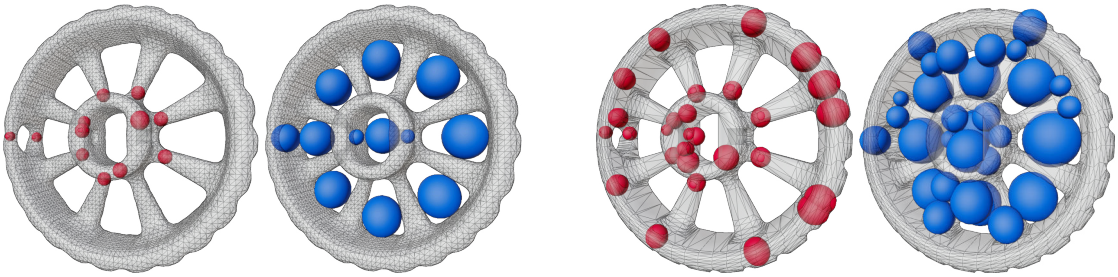


Figure 5.19 – 1-dimensional T-balls (first image), 1-dimensional B-balls (second image) of a *gear* obtained from an appropriate sampling. 1-dimensional T-balls (third image), 1-dimensional B-balls (fourth image) of a *gear* obtained from a bad sampling.

Unfortunately, computing an ε-sampling is challenging, and the related literature is sparse. An algorithm for such computation is proposed in BOISSONNAT et al. 2005. It relies on the computation of a lower bound of the distance to the medial axis. In practice, this lower bound is often obtained by approximating the medial axis itself, which creates a circular dependence. An alternative sampling approach is proposed in CHENG et al. 2004, which may also be suitable for our work. Another idea would be

to run the full hole-ball algorithm on an initial small sampling, and then refine it near the topologically critical points.

5.5.5. Catching the right ball centers

Using both methods with filtration defined by the sdf values at Voronoi vertices can however result in balls not located exactly where they are expected to be. This is due to the fact that some topologically critical points may not be approached by Voronoi vertices, even though they belong to the induced medial axis approximation (see Figure 5.20). This can happen even if the density of the initial sampling tends to infinity.

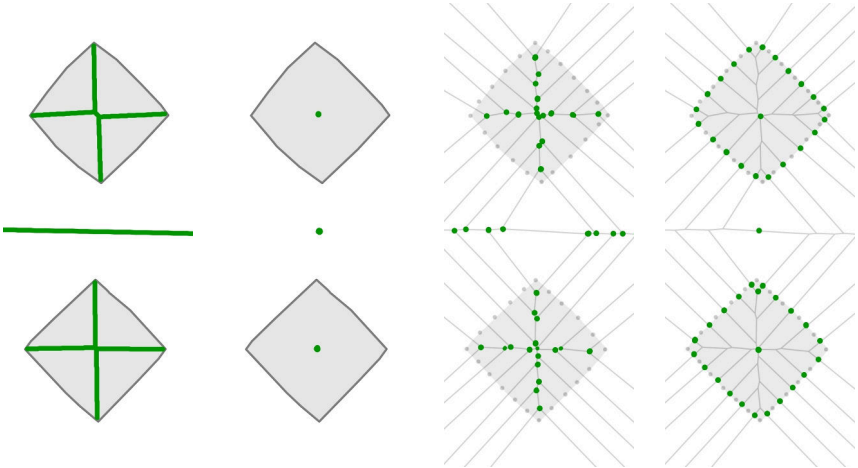


Figure 5.20 – Two squares and its medial axis in green (a), its topologically critical points (b), the Voronoi vertices of a surface sampling (c) and the Delaunay critical points of this sampling (d). The Voronoi vertices miss the central topologically critical point (c) whereas the Delaunay critical points catch all of them (d).

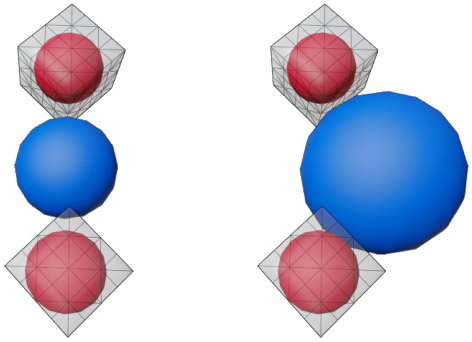


Figure 5.21 – *two-cubes* object and its 0-dimensional hole-balls obtained with (left) and without (right) taking into account Delaunay critical points.

Inspired by GIESEN, RAMOS, et al. 2011; CAZALS et al. 2008; SIERSMA 1996, we consider the following definition:

Definition 5.5.3 (Delaunay critical points)**.** Given a Delaunay triangulation of a point set P, a *Delaunay critical point* is the intersection point of a Delaunay face with its dual Voronoi face (if the intersection exists). These critical points actually correspond to topologically critical points of the discrete object P.

Despite the fact that topologically critical points are not approached by Voronoi vertices in some particular cases, we conjecture that under some regularity assumptions they can always be approached by Delaunay critical points (see Figure 5.20):

Conjecture 5.5.1. *Let X be a regular object of $\mathbb{R}^n_*$. Suppose that for all ε, P_ε is an ε-sampling of ∂X. Denote $T(P_\varepsilon)$ the Delaunay critical points induced by P_ε. Then for all topologically critical points c of X,*

$$d(c, T(P_\varepsilon)) \xrightarrow{\varepsilon \to 0} 0$$

We implemented an improvement of our methods that incorporates Delaunay critical points within the filtrations. As computing intersections between Delaunay and Voronoi cells is cheap, the extra cost is negligible. This improvement has been tested experimentally without any wrong results (see Figure 5.21). Conjecture 5.5.1 would theoretically guarantee that our implemented methods indeed approximate the hole measures, as increasing the sampling precision would make the balls converge to the true hole-balls.

5.5.6. Complexity

The complexity of the algorithm highly depends on the initial sampling used to compute the Voronoi approximation: the complexity of the persistence algorithm is $O(m^3)$ where m is the number of cells in the filtration (see H. EDELSBRUNNER and HARER 2008). In our methods, m is bounded by the size of the Voronoi approximation of the medial axis. Fortunately, if the sampling of size N is uniform enough, it is known that the computation and the size of the Voronoi diagram is linear in N (see AMENTA, ATTALI, et al. 2007; DWYER 1991). This implies that the theoretical complexity in time of our two methods is $O(N^3)$.

Fortunately, with a sparse matrix implementation, the persistence algorithm takes less than a cubic time in practice: as pointed out in H. EDELSBRUNNER and HARER 2008 the running time is at most proportional to the sum of squares of hole lifetimes. This gives a total complexity of $O(\beta N^2)$ where β is the number of holes and N the number of samples.

It is worth noting that the medial axes approach is theoretically faster. Indeed, the Voronoi filtration has a lot more cells than the discrete medial axes filtrations combined (about the number of Delaunay cells that belong to the object boundary). This is the counterpart of only computing a partial persistence diagram.

3D model	β	N	Medial axes approach			Link approach		
			Total size of filtrations	Filtration time (s)	Persistence time (s)	Size of filtration	Filtration time (s)	Persistence time (s)
gear	13	11 182	261 406	24.92	2.94	321 825	16.81	10.40
four-components	7	6 536	145 166	12.47	1.70	181 405	8.35	5.69
hollow-cubic-ball	6	5 434	108 674	9.74	1.57	126 513	6.83	4.19
triakis-tetrahedron	12	3 745	81 178	6.42	1.32	100 751	4.16	3.05
knot	2	3 200	74 896	6.02	1.19	91 325	4.13	2.83
egg-in-a-box	3	1 727	27 184	2.63	0.70	32 011	1.62	1.01
two-rings	4	1 000	22 564	1.54	0.64	26 473	1.08	1.02
eight	3	315	7 396	0.51	0.29	8 699	0.33	0.42
two-cubes	2	196	2 300	0.25	0.11	2 541	0.20	0.25

Table 5.1 – Summary of the experiments, comparing number of holes (β), sampling size (N), filtration size (total number of elements in the inner and outer medial filtrations combined, and number of elements in the Voronoi filtration), running time for the construction of the filtration(s), and running time for the persistent homology computation.

Table 5.1 aggregates the running times of our implementation of the two approaches. As expected the persistence running time in the medial axes approach is lower than in the link approach (about 3 times lower). Nevertheless, concerning the filtration(s) construction, the medial axes approach is about 1.5 slower than the link approach, due to our current implementation. We believe there is room for optimization in our implementation, especially regarding the filtration(s) construction step, which is currently much slower than the persistence step using PHAT.

5.6. Prospects: Defining and Computing Hole Shapes

This small section briefly presents ongoing research that aims at defining and computing the "geometrical shape" of a hole, using the *distance field flow complex*. We focus on the main ideas and intuitions, and omit technical subtleties.

Let X be an object in $\mathbb{R}^n$. We denote by df the unsigned distance function that assigns to each point in $\mathbb{R}^n$ its distance to the boundary of X. This function is not smooth, as its gradient is not well defined on the medial axis. However, it is possible to define a generalized notion of gradient on the medial axis that extends the "steepest ascent" property of the classical gradient (see for instance GIESEN, RAMOS, et al. 2011; SONG et al. 2025; T. K. DEY, GIESEN, et al. 2005).

We proceed as follows. Given a point $x \in \mathbb{R}^n$, we define its *driver* $d(x)$ as the center of the smallest ball containing $A_X(x)$. Then, the generalized gradient of df is

$$\nabla df(x) = \frac{x - d(x)}{||x - d(x)||}$$

and is zero whenever $x = d(x)$.

Points x such that $\nabla d_f(x) = 0$ are called *critical points*. In general, such points do not necessarily coincide with topologically critical points. However, it has been showed in SONG et al. 2025 that for generic embeddings of manifolds, these two notions coincide. In these generic settings, critical points are isolated and non-degenerate and we can use standard results of Morse theory. In particular, every critical point x is associated with a *Morse index* λ_x, roughly representing in how many directions this point is a maximum.

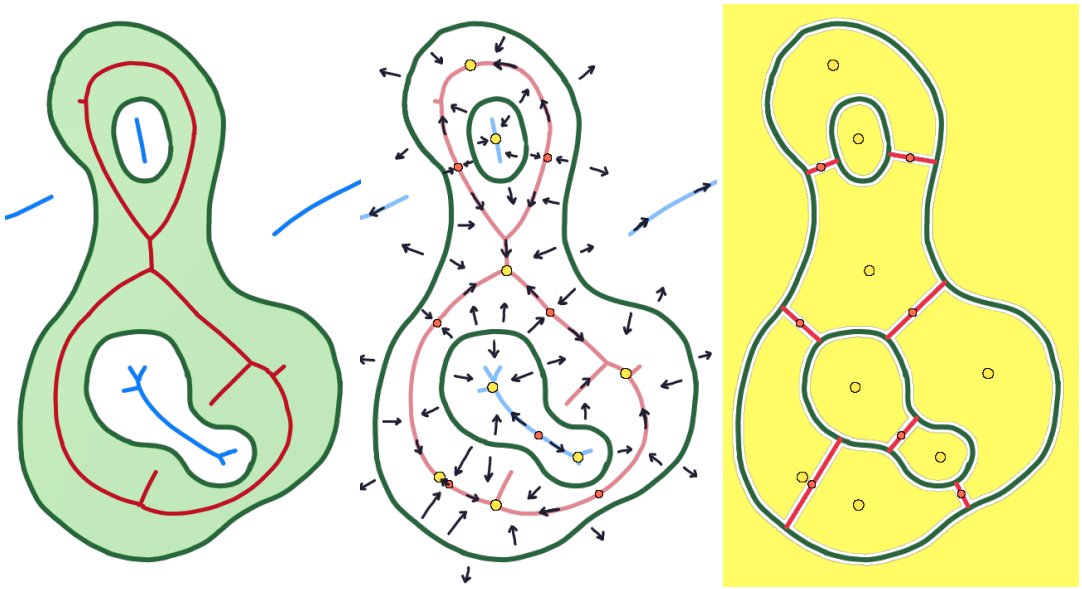


Figure 5.22 – A 2D object and its medial axis (top left). The top-right image represents the d_f-flow and its critical points, which are colored according to their Morse index (red for index 1, yellow for index 2). The bottom image is the resulting d_f-flow complex, with cells colored according to their corresponding critical points.

The generalized gradient induces a flow, called the *d_f-flow* which intuitively moves points away from the boundary. By analyzing the asymptotic behavior of this flow, we can define the associated *flow complex* (see GIESEN and JOHN 2008; CAZALS et al. 2008).

Definition 5.6.1 (flow complex)**.** We associate to each critical point x the set of all points that flow toward it, when flowing to infinity. This set is called the *d_f-flow cell* of x and is denoted by c_x.

Under non-degeneracy assumptions, every c_x and is homeomorphic to a ball of dimension λ_x: the dimension of the cell coincides with the Morse index. Moreover, the set of all d_f-flow cells defines a CW-complex structure on $\mathbb{R}^n$, called the *d_f-flow complex*. See Figure 5.22 for an illustration.

This flow complex admits a natural filtration: each flow cell c_x arrives at time $d_{f_X}(x)$. This filtration is related to the classical d_f-filtration, but it directly encodes the locations and times at which topological changes occur. We conjecture that the

persistence of this filtration recovers the persistence of the d_f-filtration of X. Such a result could improve the efficiency of our methods for computing hole balls.

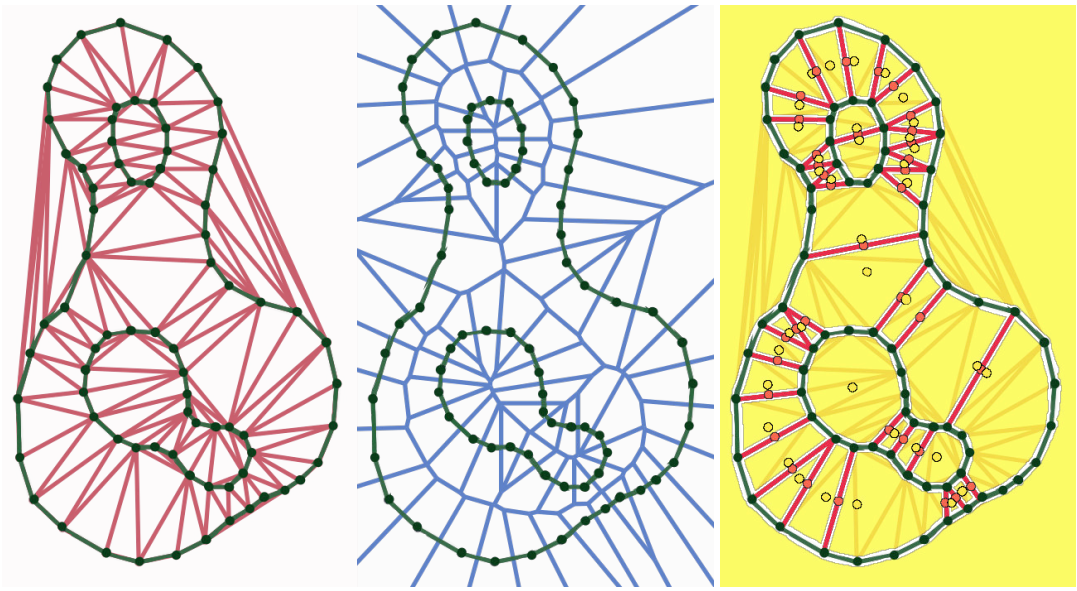


Figure 5.23 – A Delaunay triangulation (top left), its dual Voronoi diagram (top right) computed from a boundary sampling, and the resulting d_f-flow complex approximation (bottom). The Delaunay critical points (intersections of Delaunay cells with their dual Voronoi cell) of index 1 and 2 are colored in red and yellow, as well as their corresponding cells. Critical points that lie on the boundary are discarded. Although this discrete complex has more noise than the ideal continuous one in Figure 5.22, we expect that their persistence diagrams are close and that we could filter non relevant topological artifacts.

Moreover, we conjecture that the flow complex can be approximated from a finite sampling using the Delaunay triangulation. In this discrete case, critical points correspond to intersections between Delaunay cells and their dual Voronoi cells. The associated flow cells can then be approximated with unions of Delaunay cells. See Figure 5.23 for an illustration.

This formalism could also provide geometric information about homological holes. In particular, flow cells tend to fill holes in accordance with their geometry. This approach is particularly valuable in dimension three and higher, see Figure 5.24. This observation motivates the definition of a notion of "hole shape" based on the d_f-flow complex.

A possible idea would be to associate to each critical point a union of neighboring flow cells. The shape of a q-hole would then be represented by a $(q+1)$-dimensional surface that fills the hole while respecting the geometry of the object.

We aim to develop a theoretical definition of hole shape, with maybe stability properties. A challenge is to propose a definition in the general settings, where critical points can be degenerate and non isolated. A promising idea is to use the notions

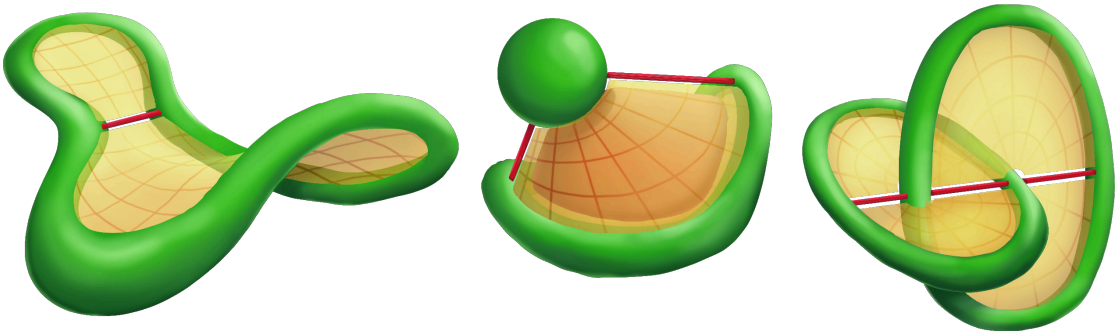


Figure 5.24 – Examples of outer df-flow complexes for three 3D shapes. Infinite flow cells are omitted for clarity. Interestingly, 2-cells fill 1-dimensional holes according to the flow and following the geometry of the shape.

of *Conley index* and *Conley complex*. We also aim to build algorithms to compute or approximate these shapes for discrete data. These works constitute the subject of a master degree research internship, conducted by Madhav CHERUPILIL SAJEEV.

5.7. Conclusion

We developed new methods to compute geometric measures of holes in volumetric shapes. In addition we implemented it for 3D objects represented by a surface mesh. Our approach is based on persistent homology and uses the notion of medial axis, which provides a powerful link between geometry and topology. Both of our methods intend to compute every hole-ball of the analyzed shape, and one of both also provides the pairing between thickness-balls and breadth-balls. An advantage of our approach is that it directly focuses on the part that carries the information we need: the medial axis. From a complexity point of view, this is significant because the computation of hole measures does not require an expensive triangulation of the space but simply a discretization of the medial axes.

From a theoretical point of view, we extended the definition of hole measure to arbitrary volumetric objects and put forward the theoretical links between geometry and topology that are provided by medial axes and persistent homology. Yet, to fully ensure the correctness of our methods, some conjectures about algebraic topology and discrete geometry still remain to be proved (see Conjectures 5.4.1 and 5.5.1).

Future work and prospects

As highlighted in Section 5.5.4, the main practical challenge of our approach is to have a sufficiently precise sampling. A future work would be to integrate an appropriate sampling algorithm in the pipeline, so that the user only needs to provide a mesh of the boundary of the object. More generally, more robust or theoretically founded medial axis approximations could be used. Potential candidates include the *core* GIESEN, RAMOS, et al. 2011 and the approximation proposed in WANG et al. 2024.

From a validation perspective, it would be relevant to compute hole-balls on CAD models of the ABC dataset KOCH et al. 2019 and compare them to the known geometrical parameters of their holes. Intuitively, the hole measure should be consistent with these parameters. This could be used as a formal topological method in CAD reverse engineering.

Besides our current ongoing work about the definition and computation of *hole shapes*, another relevant geometric hole descriptor is the notion of *minimal generators*. Minimal generators are homology generators that minimize a quantity (typically a length, area or volume depending on the dimension of the cycle), see for instance OBAYASHI 2017; T. DEY et al. 2020. The extended notion of *minimal persistent generators* was introduced in T. K. DEY, HOU, et al. 2019. They correspond to minimal generators along a filtration and can be associated to points in a persistent diagram. It would be interesting to put in parallel minimal persistent cycles, homology quartets (see Section 4.2.6), hole-balls, and hole shapes, as they all provide complementary representations of topological holes.

Conclusion

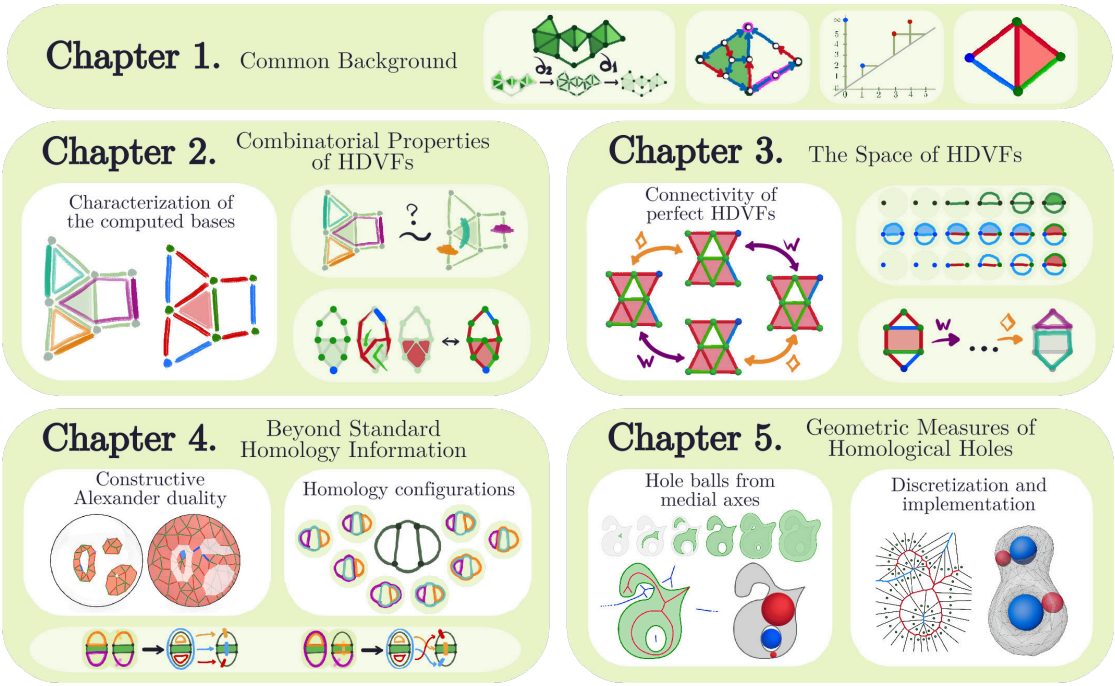


Figure 5.1 – Illustrations of the chapters and contributions of this manuscript.

Apart from the detailed conclusions of each chapter, we summarize here the three global contributions of this thesis.

HDVFs as a theoretical framework to study the combinatorics of homology computations

This thesis highlights that homology computations are not purely algebraic but also inherently combinatorial. Homological discrete vector fields (HDVFs) provide a relevant framework for studying homology computations from this combinatorial point of view, as they encompass several standard computational homology methods (Smith normal form, discrete Morse theory, persistent homology).

In Chapter 2, we prove that perfect HDVFs are equivalent to tri-partitions, another computational homology structure defined from different combinatorial concepts (trees and cotrees). This equivalence yields a deeper mutual understanding of both structures.

We further show that the homology and cohomology bases obtained by HDVFs, and by related methods, are constrained by combinatorics. More precisely, these

bases coincide exactly with *explicit* bases, a class of bases that generalize the notion of strictly fundamental bases in graph theory.

The HDVF framework also leads to constructive versions of classical homology morphisms. In Chapter 4, we introduce a constructive version of Alexander duality based on HDVFs. This particular construction also highlights that non-perfect HDVFs, which correspond to partial homology computations, are also relevant for theoretical purposes. More generally, these constructions raise the question of whether a given homology morphism is constructible by an HDVF.

Several research directions emerge from this combinatorial point of view:

- *Extending this combinatorial framework to related subjects of computational topology and homological algebra, including zigzag persistence, multidimensional persistence, combinatorial Laplacians, Conley complexes, and chain homotopies.*
- *Investigating the role of combinatorics in practical applications that rely on homology computations, for instance hole-filling or hole-opening methods based on homology generators.*
- *Extending HDVFs to homology over $\mathbb{Z}$. This represents a major challenge, as the greedy completion of HDVFs no longer holds, whereas it is a central ingredient of the theory.*

Studying homology computations from global points of view

This thesis investigates homology computations from multiple global perspectives.

In Chapter 3, we refine the relationship between HDVFs and persistent homology. We show that every filtration induces a unique perfect HDVF and that every perfect HDVF is induced by at least one filtration. Therefore, perfect HDVFs provide a higher-level representation of filtration spaces and their associated homology computations. More generally, our results suggest the following hierarchy:

filtrations → perfect HDVFs and tri-partitions → explicit bases → homology configurations

In addition, HDVFs can be organized into a graph whose vertices are HDVFs and whose edges correspond to elementary swapping operations. Chapter 3 investigates this graph both theoretically and experimentally. In particular, we provide a constructive proof of the connectedness of the graph of perfect HDVFs, and develop an algorithm for navigating this graph to adapt a given HDVF to a given explicit basis.

From a higher point of view, homology configurations, introduced in Chapter 4, classify perfect HDVFs, explicit bases, and tri-partitions. They reveal additional properties of homology computations. For instance, homology and cohomology configurations coincide, highlighting the intricate relationship between homology and cohomology computations. Homology configurations also provide a framework for characterizing the homological morphisms constructible by HDVFs and related methods. Remarkably, configurations of maximal dimension define a combinatorial invariant, and thus a topological invariant for complexes of dimension at most three. This illustrates how the study of homology computations can reveal additional topological information.

These global points of view raise several perspectives:

- *Investigating further (both experimentally and theoretically) the global structure of the graph of HDVFs.*
- *Refining our understanding of the interactions between the different levels of the hierarchy, in particular the set of filtrations associated to a given HDVF or configuration.*
- *Exploring alternative descriptions of the space of homology computations, for instance through vines and vineyards or through matroid theory.*
- *Finally, homology configurations open the broader question of combinatorial invariants. It would be interesting to compare different invariants and study their discriminative power. It would also be relevant to investigate what happens to them in dimensions greater than three, where the Hauptvermutung no longer applies.*

Representations of homological holes

A recurring theme of this thesis is the representation of homological holes.

The classical way of representing holes is with homology or cohomology generators and bases. In Chapter 4, we introduce a complementary representation, namely the *homology quartets*, based on our constructive version of Alexander duality. They consist of four chains that intuitively surround, fill, traverse, and cut a hole.

In Chapter 5, we propose a geometric representation based on persistent homology. This representation consists of two balls: the *breadth ball*, corresponding to the largest ball contained in a hole, and the *thickness ball*, corresponding to the smallest ball capable of opening it. We introduce two theoretical methods to compute them, based on the medial axis. We then implement discrete approximations of these constructions and discuss the associated algorithmic challenges.

These works raise several future perspectives:

- *Addressing the sampling issues related to the discretizations of the methods of Chapter 5.*
- *Defining and computing a geometric notion of hole shape. A challenge is to obtain a definition that remains valid in the presence of degeneracies.*
- *Studying the classical problem of minimal generators and minimal bases, from the perspective of explicit bases and homology configurations.*
- *Finally, it would be relevant to aggregate the representations introduced in this thesis, both to compare them and to provide richer geometric-topological descriptors.*

Bibliography

[Ala+15] Sylvie ALAYRANGUES, Guillaume DAMIAND, Pascal LIENHARDT, et al. "Homology of cellular structures allowing multi-incidence". In: *Discrete & Computational Geometry* 54.1 (2015), pp. 42–77 (cit. on p. 37).

[Ala+09] Sylvie ALAYRANGUES, Samuel PELTIER, Guillaume DAMIAND, et al. "Border operator for generalized maps". In: *International Conference on Discrete Geometry for Computer Imagery*. Springer. 2009, pp. 300–312 (cit. on p. 37).

[Ale24] P. ALEXANDROFF. "Über die Metrisation der im Kleinen kompakten topologischen Räume". In: *Mathematische Annalen* 92 (1924), pp. 294–301 (cit. on p. 181).

[All+19] Madjid ALLILI, Tomasz KACZYNSKI, Claudia LANDI, et al. "Acyclic partial matchings for multidimensional persistence: Algorithm and combinatorial interpretation". In: *Journal of Mathematical Imaging and Vision* 61.2 (2019), pp. 174–192 (cit. on p. 45).

[AAD07] Nina AMENTA, Dominique ATTALI, and Olivier DEVILLERS. "Complexity of Delaunay Triangulation for Points on Lower-Dimensional Polyhedra". In: *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '07. USA: SIAM, 2007, pp. 1106–1113. ISBN: 9780898716245 (cit. on p. 203).

[AB98] Nina AMENTA and Marshall BERN. "Surface Reconstruction by Voronoi Filtering". In: *Proceedings of the Fourteenth Annual Symposium on Computational Geometry*. SCG '98. Minneapolis, Minnesota, USA: Association for Computing Machinery, 1998, pp. 39–48. ISBN: 0897919734. DOI: 10.1145/276884.276889. URL: https://doi.org/10.1145/276884.276889 (cit. on p. 183).

[ACK00] Nina AMENTA, Sunghee CHOI, and Ravi KOLLURI. "The Power Crust, Unions of Balls, and the Medial Axis Transform". In: *Computational Geometry* 19 (Aug. 2000). DOI: 10.1016/S0925-7721(01)00017-7 (cit. on pp. 196, 198, 201).

[And+26] Menjatiana ANDRIANABINITSOA, Alexandra BAC, Yann-Situ GAZULL, et al. "Experiments on the Space of Homology Computations". 2026 (cit. on pp. 104, 105).

[ABE09] Dominique ATTALI, Jean-Daniel BOISSONNAT, and Herbert EDELSBRUNNER. "Stability and Computation of Medial Axes: a State-of-the-Art Report". In: *Mathematical Foundations of Scientific Visualization, Computer Graphics, and Massive Data Exploration* (Jan. 2009). DOI: 10.1007/b106657_6 (cit. on p. 196).

[BJ24] Archana BABU and Sunil JOHN. "Persistent homology based Bottleneck distance in hypergraph products". In: *Applied Network Science* 9 (Apr. 2024). DOI: 10.1007/s41109-024-00617-3 (cit. on p. 49).

[Bac26] Alexandra BAC. "Homological Discrete Vector Fields". In: *CGAL User and Reference Manual*. 6.2. CGAL Editorial Board, 2026. URL: https://cgal.github.io/9004/v0/HDVF/index.html#Chapter_HDVF (cit. on pp. 104, 114).

[Bau+14] Ulrich BAUER, Michael KERBER, Jan REININGHAUS, et al. "PHAT – Persistent Homology Algorithms Toolbox". In: *Mathematical Software – ICMS 2014*. Ed. by Hoon HONG and Chee YAP. Berlin, Heidelberg: Springer, 2014, pp. 137–143. ISBN: 978-3-662-44199-2 (cit. on p. 198).

[BB20] Mattia BERGOMI and Adriano BARATÈ. "Homological persistence in time series: an application to music classification". In: *Journal of Mathematics and Music* 14 (July 2020), pp. 1–18. DOI: 10.1080/17459737.2020.1786745 (cit. on pp. 46, 98, 113).

[BF23] Mattia G. BERGOMI and Massimo FERRI. "Exploring Graph and Digraph Persistence". In: *Algorithms* 16.10 (2023). ISSN: 1999-4893. DOI: 10.3390/a16100465. URL: https://www.mdpi.com/1999-4893/16/10/465 (cit. on p. 49).

[BFT23] Mattia G. BERGOMI, Massimo FERRI, and Antonella TAVAGLIONE. "Steady and ranging sets in graph persistence". In: *Journal of Applied and Computational Topology* 7 (2023), pp. 33–56 (cit. on p. 49).

[BV20] Mattia G. BERGOMI and Pietro VERTECHI. "Rank-based persistence". In: *Theory and Applications of Categories* 35(9) (2020), pp. 228–260 (cit. on p. 49).

[Ber25] Gilles BERTRAND. "Morse Sequences: A Simple Approach to Discrete Morse Theory". In: *Journal of Mathematical Imaging and Vision* 67 (Mar. 2025). DOI: 10.1007/s10851-025-01232-2 (cit. on pp. 44, 45).

[Bia+08] Silvia BIASOTTI, Andrea CERRI, Patrizio FROSINI, et al. "Multidimensional Size Functions for Shape Comparison". In: *Journal of Mathematical Imaging and Vision* 32 (Oct. 2008), pp. 161–179. DOI: 10.1007/s10851-008-0096-z (cit. on p. 48).

[BM88] Edward BIERSTONE and Pierre D MILMAN. "Semianalytic and subanalytic sets". In: *Publications Mathématiques de l'IHÉS* 67 (1988), pp. 5–42 (cit. on p. 185).

[BT08] Anders BJÖRNER and Martin TANCER. "Note: Combinatorial Alexander Duality—A Short and Elementary Proof". In: *Discrete & Computational Geometry* 42.4 (Aug. 2008), pp. 586–593. ISSN: 1432-0444. DOI: 10.1007/s00454-008-9102-x. URL: http://dx.doi.org/10.1007/s00454-008-9102-x (cit. on p. 117).

[BO05] Jean-Daniel BOISSONNAT and Steve OUDOT. "Provably good sampling and meshing of surfaces". In: *Graphical Models* 67.5 (2005). Solid Modeling and Applications, pp. 405–451. ISSN: 1524-0703. DOI: https://doi.org/10.1016/j.gmod.2005.01.004 (cit. on p. 201).

[BNG22] Nicolas BOUTRY, Laurent NAJMAN, and Thierry GÉRAUD. "Some equivalence relation between persistent homology and morphological dynamics". In: *Journal of Mathematical Imaging and Vision* 64 (June 2022), pp. 807–824. DOI: 10.1007/s10851-022-01104-z. URL: https://hal.science/hal-03676854 (cit. on p. 46).

[Bre+16] Stéphane BRESSAN, Jingyan LI, Shiquan REN, et al. *The embedded homology of hypergraphs and applications*. 2016. URL: https://api.semanticscholar.org/CorpusID:119141581 (cit. on p. 49).

[BAK24] Guillaume BROUILLETTE, Madjid ALLILI, and Tomasz KACZYNSKI. "Multiparameter discrete Morse theory". In: *Journal of Applied and Computational Topology* 8.7 (2024), pp. 2155–2196 (cit. on p. 45).

[Bro69] Edward M. BROWN. "The Hauptvermutung for 3-Complexes". In: *Transactions of the American Mathematical Society* 144 (1969), pp. 173–196. ISSN: 00029947. URL: http://www.jstor.org/stable/1995276 (visited on 07/03/2025) (cit. on p. 162).

[BS14] Peter BUBENIK and Jonathan A. SCOTT. "Categorification of Persistent Homology". In: *Discrete and Computational Geometry* 51.3 (Jan. 2014), pp. 600–627. DOI: 10.1007/s00454-014-9573-x. URL: https://doi.org/10.1007%2Fs00454-014-9573-x (cit. on pp. 46, 48).

[BSS14] Peter BUBENIK, Vin de SILVA, and Jonathan SCOTT. "Metrics for Generalized Persistence Modules". In: *Foundations of Computational Mathematics* 15.6 (Oct. 2014), pp. 1501–1531. ISSN: 1615-3383. DOI: 10.1007/s10208-014-9229-5 (cit. on p. 48).

[CZ07] Gunnar CARLSSON and Afra ZOMORODIAN. "The Theory of Multidimensional Persistence". In: *Discrete and Computational Geometry* 42 (June 2007), pp. 71–93. DOI: 10.1007/s00454-009-9176-0 (cit. on p. 48).

[CS08] Gunnar E. CARLSSON and Vin de SILVA. "Zigzag Persistence". In: *Foundations of Computational Mathematics* 10 (2008), pp. 367–405. URL: https://api.semanticscholar.org/CorpusID:6668405 (cit. on p. 48).

[Car+20] Mathieu CARRIÈRE, Frédéric CHAZAL, Yuichi IKE, et al. "Perslay: A neural network layer for persistence diagrams and new graph topological signatures". In: *International Conference on Artificial Intelligence and Statistics*. PMLR. 2020, pp. 2786–2796 (cit. on p. 46).

[CPP08] Frédéric CAZALS, Aditya PARAMESWARAN, and Sylvain PION. "Robust Construction of the Three-Dimensional Flow Complex". In: *ACM Symposium on Computational Geometry (SCG)*. Washington, United States, June 2008. URL: https://hal.inria.fr/inria-00344962 (cit. on pp. 194, 196, 203, 205).

[CV15] Erin Wolf CHAMBERS and Mikael VEJDEMO-JOHANSSON. "Computing minimum area homologies". In: *Computer graphics forum*. Vol. 34. 6. Wiley Online Library. 2015, pp. 13–21 (cit. on p. 180).

[Cha+09a] Frédéric CHAZAL, David COHEN-STEINER, Marc GLISSE, et al. "Proximity of persistence modules and their diagrams". In: SCG '09. Aarhus, Denmark: Association for Computing Machinery, 2009, pp. 237–246. ISBN: 9781605585017. DOI: 10.1145/1542362.1542407 (cit. on p. 47).

[Cha+09b] Frédéric CHAZAL, David COHEN-STEINER, Leonidas GUIBAS, et al. "Gromov-Hausdorff Stable Signatures for Shapes using Persistence". In: *Comput. Graph. Forum* 28 (July 2009), pp. 1393–1403. DOI: 10.1111/j.1467-8659.2009.01516.x (cit. on p. 180).

[CL05] Frédéric CHAZAL and André LIEUTIER. "The λ-medial axis". In: *Graphical Models graphical Models and Image Processing computer Vision, Graphics, and Image Processing* 67 (2005), pp. 304–331 (cit. on pp. 191, 194, 196, 198).

[CL06] Frédéric CHAZAL and André LIEUTIER. "Topology guaranteeing manifold reconstruction using distance function to noisy data". In: vol. 2006. June 2006, pp. 112–118. DOI: 10.1145/1137856.1137876 (cit. on p. 183).

[CS04] Frédéric CHAZAL and R. SOUFFLET. "Stability and Finiteness Properties of Medial Axis and Skeleton". In: *Journal of Dynamical and Control Systems* 10 (Apr. 2004), pp. 149–170. DOI: 10.1023/B:JODS.0000024119.38784.ff (cit. on p. 185).

[CK11] Chao CHEN and Michael KERBER. "Persistent homology computation with a twist". In: *Proceedings 27th European workshop on computational geometry*. Vol. 11. 2011, pp. 197–200 (cit. on p. 48).

[CDR04] Siu-Wing CHENG, Tamal K. DEY, and Edgar A. RAMOS. "Sampling and Meshing a Surface with Guaranteed Topology and Geometry". In: *Proceedings of the 20th Annual Symposium on Computational Geometry*. 2004 (cit. on p. 201).

[CD23] Mattéo CLÉMOT and Julie DIGNE. "Neural skeleton: Implicit neural representation away from the surface". In: *Computers & Graphics* 114 (2023), pp. 368–378 (cit. on p. 197).

[CDT25] Mattéo CLÉMOT, Julie DIGNE, and Julien TIERNY. "Topological Autoencoders++: Fast and Accurate Cycle-Aware Dimensionality Reduction". In: *IEEE Transactions on Visualization and Computer Graphics* (2025) (cit. on p. 46).

[CEH05] David COHEN-STEINER, Herbert EDELSBRUNNER, and John HARER. "Stability of Persistence Diagrams". In: *Proceedings of the Twenty-First Annual Symposium on Computational Geometry*. SCG '05. Pisa, Italy: Association for Computing Machinery, 2005, pp. 263–271. ISBN: 1581139918. DOI: 10.1145/1064092.1064133 (cit. on p. 47).

[Coh+10] David COHEN-STEINER, Herbert EDELSBRUNNER, John HARER, and Yuriy MILEYKO. "Lipschitz functions have L p-stable persistence". In: *Foundations of computational mathematics* 10.2 (2010), pp. 127–139 (cit. on p. 47).

[CEM06] David COHEN-STEINER, Herbert EDELSBRUNNER, and Dmitriy MOROZOV. "Vines and vineyards by updating persistence in linear time". In: *Proceedings of the 22nd ACM Symposium on Computational Geometry*. Ed. by Nina AMENTA and Otfried CHEONG. ACM, 2006, pp. 119–126. DOI: 10.1145/1137856.1137877. URL: https://doi.org/10.1145/1137856.1137877 (cit. on pp. 98, 113).

[ČV23] Matija ČUFAR and Žiga VIRK. "Fast computation of persistent homology representatives with involuted persistent homology". In: *Foundations of Data Science* 5.4 (2023), pp. 466–479. ISSN: 2639-8001. DOI: 10.3934/fods.2023006 (cit. on p. 84).

[CKM04] Tim CULVER, John KEYSER, and Dinesh MANOCHA. "Exact Computation of the Medial Axis of a Polyhedron". In: *Comput. Aided Geom. Des.* 21.1 (Jan. 2004), pp. 65–98. ISSN: 0167-8396. DOI: 10.1016/j.cagd.2003.07.008. URL: https://doi.org/10.1016/j.cagd.2003.07.008 (cit. on p. 196).

[CS25] Carina CURTO and Nicole SANDERSON. "Topological neuroscience: linking circuits to function". In: *Annual Review of Neuroscience* 48 (2025) (cit. on p. 46).

[DFL08] Michele D'AMICO, Patrizio FROSINI, and Claudia LANDI. "Natural Pseudo-Distance and Optimal Matching between Reduced Size Functions". In: *Acta Applicandae Mathematicae* 109 (2008), pp. 527–554. URL: https://api.semanticscholar.org/CorpusID:1704971 (cit. on p. 47).

[DLW25] Hana DAL POZ KOUŘIMSKÁ, André LIEUTIER, and Mathijs WINTRAECKEN. "The medial axis of any closed bounded set is locally Lipschitz stable with respect to the Hausdorff distance under ambient diffeomorphisms". In: *Journal of Applied and Computational Topology* 9.3 (2025), p. 20 (cit. on p. 185).

[Dam26] Guillaume DAMIAND. "Linear Cell Complex". In: *CGAL User and Reference Manual*. 6.1.1. CGAL Editorial Board, 2026. URL: https://doc.cgal.org/6.1.1/Manual/packages.html#PkgLinearCellComplex (cit. on p. 27).

[DE95] Cecil Jose A. DELFINADO and Herbert EDELSBRUNNER. "An incremental algorithm for Betti numbers of simplicial complexes on the 3-sphere". In: *Comput. Aided Geom. Des.* 12.7 (1995), pp. 771–784. DOI: 10.1016/0167-8396(95)00016-Y. URL: https://doi.org/10.1016/0167-8396(95)00016-Y (cit. on pp. 41, 116).

[DPK82] Narsingh DEO, Gurpur M. PRABHU, and Mukkai S. KRISHNAMOORTHY. "Algorithms for Generating Fundamental Cycles in a Graph". In: *ACM Transactions on Mathematical Software (TOMS)* 8 (1982), pp. 26–42. URL: https://api.semanticscholar.org/CorpusID:2260051 (cit. on pp. 84, 167).

[DHM20] Tamal DEY, Tao HOU, and Sayan MANDAL. "Computing Minimal Persistent Cycles: Polynomial and Hard Cases". In: Jan. 2020, pp. 2587–2606. ISBN: 978-1-61197-599-4. DOI: 10.1137/1.9781611975994.158 (cit. on pp. 83, 208).

[DFW13] Tamal K. DEY, Fengtao FAN, and Yusu WANG. "An Efficient Computation of Handle and Tunnel Loops via Reeb Graphs". In: *ACM Trans. Graph.* 32.4 (July 2013). ISSN: 0730-0301. DOI: 10.1145/2461912.2462017. URL: https://doi.org/10.1145/2461912.2462017 (cit. on p. 139).

[Dey+05] Tamal K. DEY, Joachim GIESEN, Edgar A. RAMOS, et al. "Critical Points of the Distance to an Epsilon-Sampling of a Surface and Flow-Complex-Based Surface Reconstruction". In: *Proceedings of the Twenty-First Annual Symposium on Computational Geometry*. SCG '05. New York, NY, USA: Association for Computing Machinery, 2005, pp. 218–227. ISBN: 1581139918. DOI: 10.1145/1064092.1064126. URL: https://doi.org/10.1145/1064092.1064126 (cit. on pp. 183, 204).

[DHM19] Tamal K. DEY, Tao HOU, and Sayan MANDAL. "Persistent 1-Cycles: Definition, Computation, and Its Application". In: *Computational Topology in Image Context*. Ed. by Rebeca MARFIL, Mariletty CALDERÓN, Fernando DÍAZ DEL RÍO, et al. Cham: Springer International Publishing, 2019, pp. 123–136. ISBN: 978-3-030-10828-1 (cit. on pp. 93, 180, 208).

[Dey+08] Tamal K. DEY, Kuiyu LI, Jian SUN, et al. "Computing Geometry-Aware Handle and Tunnel Loops in 3D Models". In: *ACM Trans. Graph.* 27.3 (Aug. 2008), pp. 1–9. ISSN: 0730-0301. DOI: 10.1145/1360612.1360644. URL: https://doi.org/10.1145/1360612.1360644 (cit. on p. 180).

[DZ04] Tamal K. DEY and Wulue ZHAO. "Approximate medial axis as a Voronoi subcomplex". In: *Computer-Aided Design* 36.2 (2004). Solid Modeling and Applications, pp. 195–202. ISSN: 0010-4485. DOI: https://doi.org/10.1016/S0010-4485(03)00061-7 (cit. on p. 196).

[DKL25] Ioannis DIAMANTIS, Louis H KAUFFMAN, and Sofia LAMBROPOULOU. "Topology and Algebra of Bonded Knots and Braids". In: *Mathematics* 13.20 (2025), p. 3260 (cit. on p. 164).

[Dło+11] Paweł DŁOTKO, Tomasz KACZYNSKI, Marian MROZEK, et al. "Coreduction homology algorithm for regular CW-complexes". In: *Discrete & Computational Geometry* 46.2 (2011), pp. 361–388 (cit. on p. 43).

[Dor+20] Harish DORAISWAMY, Julien TIERNY, Paulo JS SILVA, et al. "Topomap: A 0-dimensional homology preserving projection of high-dimensional data". In: *IEEE Transactions on Visualization and Computer Graphics* 27.2 (2020), pp. 561–571 (cit. on p. 46).

[Dwy91] R.A. DWYER. "Higher-Dimensional Voronoi Diagrams in Linear Expected Time." In: *Discrete and computational geometry* 6.4 (1991), pp. 343–368. URL: http://eudml.org/doc/131163 (cit. on p. 203).

[EH08] Herbert EDELSBRUNNER and John HARER. "Persistent homology—a survey". In: *Discrete and Computational Geometry - DCG* 453 (Jan. 2008). DOI: 10.1090/conm/453/08802 (cit. on pp. 46, 203).

[EÖ19] Herbert EDELSBRUNNER and Katharina ÖLSBÖCK. "Holes and dependences in an ordered complex". In: *Computer Aided Geometric Design* 73 (2019), pp. 1–15. ISSN: 0167-8396. DOI: https://doi.org/10.1016/j.cagd.2019.06.003 (cit. on p. 85).

[EÖ20] Herbert EDELSBRUNNER and Katharina ÖLSBÖCK. "Tri-partitions and Bases of an Ordered Complex". In: *Discrete & Computational Geometry* 64 (2020), pp. 759–775. DOI: 10.1007/s00454-020-00188-x (cit. on pp. 14, 18, 56, 63–66, 68, 85, 87, 95, 114).

[EST25] Herbert EDELSBRUNNER, Elizabeth STEPHENSON, and Martin Hafskjold THORESEN. "The Mid-sphere Cousin of the Medial Axis Transform". In: *International Conference on Discrete Geometry and Mathematical Morphology*. Springer. 2025, pp. 133–147 (cit. on p. 98).

[ELZ02] EDELSBRUNNER, LETSCHER, and ZOMORODIAN. "Topological Persistence and Simplification". In: *Discrete Comput. Geom.* 28.4 (Nov. 2002), pp. 511–533. ISSN: 0179-5376. DOI: 10.1007/s00454-002-2885-2. URL: https://doi.org/10.1007/s00454-002-2885-2 (cit. on pp. 45, 48, 63, 84, 87, 88, 113).

[EL53] Samuel EILENBERG and Saunders Mac LANE. "On the groups H (π, n), I". In: *Annals of Mathematics* 58.1 (1953), pp. 55–106 (cit. on p. 44).

[For98] Robin FORMAN. "Morse Theory for Cell Complexes". In: *Advances in Mathematics* 134.1 (1998), pp. 90–145. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.1997.1650 (cit. on pp. 41, 44, 49).

[For01] Robin FORMAN. "A User's Guide To Discrete Morse Theory". In: *Sém. Lothar. Combin.* 48 (Dec. 2001) (cit. on p. 44).

[FQ90] Michael H. FREEDMAN and Frank QUINN. *Topology of 4-Manifolds*. Princeton: Princeton University Press, 1990. ISBN: 9781400861064. DOI: doi:10.1515/9781400861064. URL: https://doi.org/10.1515/9781400861064 (cit. on p. 162).

[Fro92] Patrizio FROSINI. "Measuring shapes by size functions". In: *Intelligent robots and computer vision X: algorithms and techniques*. Vol. 1607. SPIE. 1992, pp. 122–133 (cit. on p. 45).

[Gaz25] Yann-Situ GAZULL. "Stability and extension of steady and ranging persistence". In: *Journal of Applied and Computational Topology* 10.1 (Dec. 2025). ISSN: 2367-1734. DOI: 10.1007/s41468-025-00228-6. URL: http://dx.doi.org/10.1007/s41468-025-00228-6 (cit. on pp. 16, 20, 49).

[GBG23] Yann-Situ GAZULL, Alexandra BAC, and Aldo GONZALEZ-LORENZO. "Computing Geometrical Measures of Topological Holes". In: *Computer-Aided Design* 163 (2023), p. 103563. ISSN: 0010-4485. DOI: https://doi.org/10.1016/j.cad.2023.103563 (cit. on pp. 139, 180).

[GGB23] Yann-Situ GAZULL, Aldo GONZALEZ-LORENZO, and Alexandra BAC. *Computing Geometrical Measures of Topological Holes*. https://github.com/Yann-Situ/Hole-Measures/releases/tag/paper-submission. 2023 (cit. on p. 198).

[GGB25a] Yann-Situ GAZULL, Aldo GONZALEZ-LORENZO, and Alexandra BAC. "Characterization of the Computed Homology and Cohomology Bases". In: *International Conference on Discrete Geometry and Mathematical Morphology*. Springer. 2025, pp. 163–177. DOI: 10.1007/978-3-032-09544-2_12. URL: https://dx.doi.org/10.1007/978-3-032-09544-2_12 (cit. on pp. 65, 71).

[GGB25b] Yann-Situ GAZULL, Aldo GONZALEZ-LORENZO, and Alexandra BAC. "On the Graph of Homology Computations and its Connectedness". working paper or preprint. Oct. 2025. URL: https://hal.science/hal-05763373 (cit. on p. 86).

[GJ08] Joachim GIESEN and Matthias JOHN. "The flow complex: A data structure for geometric modeling". In: *Computational Geometry* 39.3 (2008), pp. 178–190. ISSN: 0925-7721. DOI: https://doi.org/10.1016/j.comgeo.2007.01.002 (cit. on p. 205).

[GRB11] Joachim GIESEN, Edgar RAMOS, and Sadri BARDIA. "Medial Axis Approximation and Unstable Flow Complex". In: *International Journal of Computational Geometry and Applications* 18 (Nov. 2011). DOI: 10.1142/S0218195908002751 (cit. on pp. 197, 198, 201, 203, 204, 207).

[Gol02] Timothy E GOLDBERG. "Combinatorial Laplacians of simplicial complexes". PhD thesis. Bard College, 2002 (cit. on p. 85).

[Gon+11] Rocio GONZALEZ-DIAZ, Adrian ION, María JIMÉNEZ, et al. "Incremental-Decremental Algorithm for Computing AT-Models and Persistent Homology". In: *Computer Analysis of Images and Patterns*. Berlin, Heidelberg: Springer, Jan. 2011, pp. 286–293. ISBN: 978-3-642-23671-6. DOI: 10.1007/978-3-642-23672-3_35 (cit. on p. 87).

[GJM11] Rocio GONZALEZ-DIAZ, Maria Jose JIMENEZ, and Belén MEDRANO. "Cubical cohomology ring of 3D photographs". In: *International Journal of Imaging Systems and Technology* 21.1 (2011), pp. 76–85 (cit. on p. 44).

[Gon+09] Rocío GONZÁLEZ-DÍAZ, María José JIMÉNEZ, Belén MEDRANO, et al. "A tool for integer homology computation: λ-AT-model". In: *Image and Vision Computing* 27.7 (2009), pp. 837–845 (cit. on p. 44).

[Gon16] Aldo GONZALEZ-LORENZO. "Computational Homology Applied to Discrete Objects". Theses. Aix-Marseille Universite ; Universidad de Sevilla, Nov. 2016. URL: https://hal.archives-ouvertes.fr/tel-01477399 (cit. on pp. 50, 56–58, 60–63, 98, 159, 172).

[GBG24] Aldo GONZALEZ-LORENZO, Alexandra BAC, and Yann-Situ GAZULL. "A constructive approach of Alexander duality". In: *Journal of Applied and Computational Topology* 9.2 (Dec. 2024). DOI: 10.1007/s41468-024-00198-1. URL: https://doi.org/10.1007/s41468-024-00198-1 (cit. on pp. 85, 117–119).

[Gon+16a] Aldo GONZALEZ-LORENZO, Alexandra BAC, Jean-Luc MARI, et al. "Two Measures for the Homology Groups of Binary Volumes". In: *19th IAPR International Conference on Discrete Geometry for Computer Imagery (DGCI 2016)*. Ed. by SPRINGER. Vol. 9647. Apr. 2016, pp. 154–165. DOI: 10.1007/978-3-319-32360-2_12. URL: https://hal-amu.archives-ouvertes.fr/hal-01341015 (cit. on pp. 180–182).

[Gon+17] Aldo GONZALEZ-LORENZO, Alexandra BAC, Jean-Luc MARI, et al. "Allowing cycles in discrete Morse theory". In: *Topology and its Applications* 228 (2017), pp. 1–35. ISSN: 0166-8641. DOI: https://doi.org/10.1016/j.topol.2017.05.008 (cit. on pp. 45, 50, 54, 56, 60, 84, 87, 88, 113).

[Gon+16b] Aldo GONZALEZ-LORENZO, Mateusz JUDA, Alexandra BAC, et al. "Fast, Simple and Separable Computation of Betti Numbers on Three-dimensional Cubical Complexes". In: *6th International Workshop on Computational Topology in Image Context (CTIC 2016)*. Ed. by Alexandra BAC and Jean-Luc Mari (EDS.) Vol. 9667. Springer, June 2016, pp. 130–139. DOI: 10.1007/978-3-319-39441-1_12. URL: https://hal-amu.archives-ouvertes.fr/hal-01341014 (cit. on p. 41).

[Gri+20] Alexander GRIGOR'YAN, Yong LIN, Yuri V. MURANOV, et al. "Path Complexes and their Homologies". In: *Journal of Mathematical Sciences* 248 (2020), pp. 564–599 (cit. on p. 49).

[GMJ21] Alexander GRIGOR'YAN, Yuri V. MURANOV, and Rolando JIMÉNEZ. "Homology of Digraphs". In: *Mathematical Notes* 109 (2021), pp. 712–726 (cit. on p. 48).

[GMY14] Alexander GRIGOR'YAN, Yuri V. MURANOV, and Shing-Tung YAU. "Graphs associated with simplicial complexes". In: *Homology, Homotopy and Applications* 16 (2014), pp. 295–311 (cit. on p. 49).

[Haj+23] Mustafa HAJIJ, Ghada ZAMZMI, Theodore PAPAMARKOU, et al. "Combinatorial Complexes: Bridging the Gap Between Cell Complexes and Hypergraphs". In: *2023 57th Asilomar Conference on Signals, Systems, and Computers*. 2023, pp. 799–803. DOI: 10.1109/IEEECONF59524.2023.10477018 (cit. on p. 32).

[Har76] Robert HARDT. "Triangulation of subanalytic sets and proper light subanalytic maps". In: *Inventiones Mathematicae* 38 (Oct. 1976), pp. 207–217. DOI: 10.1007/BF01403128 (cit. on p. 185).

[Har+14] Shaun HARKER, Konstantin MISCHAIKOW, Marian MROZEK, et al. "Discrete Morse theoretic algorithms for computing homology of complexes and maps". In: *Foundations of Computational Mathematics* 14.1 (2014), pp. 151–184 (cit. on p. 44).

[Hat02] Allen HATCHER. *Algebraic Topology*. Algebraic Topology. Cambridge University Press and Cornell University. Department of Mathematics, 2002. ISBN: 9780521795401. URL: https://books.google.fr/books?id=BjKs86kosqgC (cit. on pp. 28, 30, 36, 39, 119, 121, 122, 135, 185, 190).

[Hua+24] Qijia HUANG, Pierre KRAEMER, Sylvain THERY, et al. "Dynamic skeletonization via variational medial axis sampling". In: *SIGGRAPH Asia 2024 Conference Papers*. 2024, pp. 1–11 (cit. on p. 197).

[IAZ25] Ekaterina S. IVSHINA, Galit ANIKEEVA, and Ling ZHOU. *Doughnut or Mickey Mouse? Detecting Toroidal Structure in Data through Persistent Cup-Length*. 2025. arXiv: 2507.11151 [math.AT]. URL: https://arxiv.org/abs/2507.11151 (cit. on p. 39).

[Jim+17] María José JIMENEZ, Matteo RUCCO, Pablo VICENTE-MUNUERA, et al. "Topological data analysis for self-organization of biological tissues". In: *International workshop on combinatorial image analysis*. Springer. 2017, pp. 229–242 (cit. on p. 46).

[JMR83] William JULIAN, Ray MINES, and Fred RICHMAN. "Alexander duality." In: *Pacific Journal of Mathematics* 106.1 (1983), pp. 115–127. DOI: pjm/1102721108. URL: https://doi.org/ (cit. on p. 116).

[KMŚ98] Tomasz KACZYŃSKI, Marian MROZEK, and Maciej ŚLUSAREK. "Homology computation by reduction of chain complexes". In: *Computers & Mathematics with Applications* 35.4 (1998), pp. 59–70 (cit. on p. 43).

[Kal13] Sara KALIŠNIK. "Alexander duality for parametrized homology". In: *Homology, Homotopy and Applications* 15.2 (2013), pp. 227–243 (cit. on pp. 195, 196).

[Koc+19] Sebastian KOCH, Albert MATVEEV, Zhongshi JIANG, et al. "ABC: A Big CAD Model Dataset For Geometric Deep Learning". In: *The IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*. June 2019 (cit. on p. 208).

[Kon+24] Jiayi KONG, Chen ZONG, Jun LUO, et al. "Quasi-Medial Distance Field (Q-MDF): A Robust Method for Approximating and Discretizing Neural Medial Axes". In: *ACM Transactions on Graphics* (2024) (cit. on p. 197).

[Koz05] Dmitry N. KOZLOV. "Discrete Morse theory for free chain complexes". In: *Comptes Rendus Mathematique* 340.12 (2005), pp. 867–872. ISSN: 1631-073X. DOI: https://doi.org/10.1016/j.crma.2005.04.036 (cit. on p. 32).

[Kur14] Vitaliy KURLIN. "Auto-completion of Contours in Sketches, Maps, and Sparse 2D Images Based on Topological Persistence". In: *16th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, SYNASC 2014, Timisoara, Romania, September 22-25, 2014*. 2014, pp. 594–601. DOI: 10.1109/SYNASC.2014.85 (cit. on pp. 46, 116).

[LS22] Claudia LANDI and Sara SCARAMUCCIA. "Relative-perfectness of discrete gradient vector fields and multi-parameter persistent homology". In: *Journal of Combinatorial Optimization* 44.4 (2022), pp. 2347–2374 (cit. on p. 45).

[Les15] Michael LESNICK. "The Theory of the Interleaving Distance on Multidimensional Persistence Modules". In: *Foundations of Computational Mathematics* 15.3 (Mar. 2015), pp. 613–650. ISSN: 1615-3383. DOI: 10.1007/s10208-015-9255-y. URL: http://dx.doi.org/10.1007/s10208-015-9255-y (cit. on p. 47).

[LR07] Christian LIEBCHEN and Romeo RIZZI. "Classes of cycle bases". In: *Discrete Applied Mathematics* 155.3 (2007), pp. 337–355. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2006.06.007 (cit. on p. 84).

[Lie03] André LIEUTIER. "Any Open Bounded Subset of $\mathbb{R}^n$ Has the Same Homotopy Type than Its Medial Axis". In: *Proceedings of the Eighth ACM Symposium on Solid Modeling and Applications*. SM '03. Seattle, Washington, USA: Association for Computing Machinery, 2003, pp. 65–75. ISBN: 1581137060. DOI: 10.1145/781606.781620. URL: https://doi.org/10.1145/781606.781620 (cit. on p. 185).

[Liu+21] Xiang LIU, Huitao FENG, Jie WU, et al. "Persistent spectral hypergraph based machine learning (PSH-ML) for protein-ligand binding affinity prediction". In: *Briefings in Bioinformatics* 22.5 (Apr. 2021), bbab127. ISSN: 1477-4054. DOI: 10.1093/bib/bbab127. URL: https://doi.org/10.1093/bib/bbab127 (cit. on p. 49).

[Liu+24] Xiang LIU, Huitao FENG, Jie WU, et al. "Computing hypergraph homology". In: *Foundations of Data Science* 6.2 (2024), pp. 172–194. DOI: 10.3934/fods.2024007 (cit. on p. 49).

[LMT22] Umberto LUPO, Anibal MEDINA-MARDONES, and Guillaume TAUZIN. "Persistence Steenrod modules". In: *Journal of Applied and Computational Topology* 6 (May 2022), pp. 1–28. DOI: 10.1007/s41468-022-00093-7 (cit. on p. 39).

[Mah+18] Jeffrey MAHLER, Florian T. POKORNY, Sherdil NIYAZ, et al. "Synthesis of Energy-Bounded Planar Caging Grasps Using Persistent Homology". In: *IEEE Trans Autom. Sci. Eng.* 15.3 (2018), pp. 908–918. DOI: 10.1109/TASE.2018.2831724 (cit. on p. 116).

[MS24] Clément MARIA and Hannah SCHREIBER. "Discrete morse theory for computing zigzag persistence". In: *Discrete & Computational Geometry* 71.2 (2024), pp. 708–737 (cit. on p. 45).

[MSZ24] Facundo MÉMOLI, Anastasios STEFANOU, and Ling ZHOU. "Persistent cup product structures and related invariants". In: *Journal of Applied and Computational Topology* 8.1 (2024), pp. 93–148 (cit. on p. 39).

[Mil61] John MILNOR. "Two Complexes Which are Homeomorphic But Combinatorially Distinct". In: *Annals of Mathematics* 74.3 (1961), pp. 575–590. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1970299 (visited on 07/03/2025) (cit. on p. 162).

[MMS11] Nikola MILOSAVLJEVIĆ, Dmitriy MOROZOV, and Primoz SKRABA. "Zigzag persistent homology in matrix multiplication time". In: *Proceedings of the twenty-seventh Annual Symposium on Computational Geometry*. 2011, pp. 216–225 (cit. on pp. 41, 48).

[MN13] Konstantin MISCHAIKOW and Vidit NANDA. "Morse theory for filtrations and efficient computation of persistent homology". In: *Discrete & Computational Geometry* 50.2 (2013), pp. 330–353 (cit. on p. 48).

[Moi52] Edwin E. MOISE. "Affine structures in 3-manifolds, V, The triangulation theorem and Hauptvermutung". In: *Annals of Mathematics* 56 (1952), pp. 96–114. URL: https://api.semanticscholar.org/CorpusID:124364659 (cit. on p. 162).

[MR12a] Helena MOLINA-ABRIL and Pedro REAL. "Homological optimality in discrete morse theory through chain homotopies". In: *Pattern Recognition Letters* 33.11 (2012), pp. 1501–1506 (cit. on p. 49).

[MR12b] Helena MOLINA-ABRIL and Pedro REAL. "Homological spanning forest framework for 2D image analysis". In: *Annals of Mathematics and Artificial Intelligence* 64.4 (2012), pp. 385–409 (cit. on p. 49).

[MB09] Marian MROZEK and Bogdan BATKO. "Coreduction homology algorithm". In: *Discrete & Computational Geometry* 41.1 (2009), pp. 96–118 (cit. on p. 43).

[Mun74] Hans J MUNKHOLM. "The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps". In: *Journal of Pure and Applied Algebra* 5.1 (1974), pp. 1–50 (cit. on p. 44).

[MKP25] James MUNKRES, Steven KRANTZ, and Harold PARKS. *Elements of Algebraic Topology*. Apr. 2025. ISBN: 9781003621478. DOI: 10.1201/9781003621478 (cit. on p. 42).

[Müt22] Torsten MÜTZE. "Combinatorial Gray codes-an updated survey". In: *arXiv preprint arXiv:2202.01280* (2022) (cit. on p. 168).

[Oba17] Ippei OBAYASHI. "Volume-Optimal Cycle: Tightest Representative Cycle of a Generator in Persistent Homology". In: *SIAM Journal on Applied Algebra and Geometry* 2 (Dec. 2017). DOI: 10.1137/17M1159439 (cit. on pp. 93, 116, 139, 208).

[OEI26] OEIS FOUNDATION INC. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org. 2026 (cit. on p. 167).

[Ott+15] Nina OTTER, Mason PORTER, Ulrike TILLMANN, et al. "A roadmap for the computation of persistent homology". In: *EPJ Data Science* 6 (June 2015). DOI: 10.1140/epjds/s13688-017-0109-5 (cit. on p. 48).

[Oud15] Steve Y Oudot. *Persistence theory: from quiver representations to data analysis*. Vol. 209. American Mathematical Society Providence, 2015 (cit. on p. 46).

[PMK23] Nicolas Passat, Julien Mendes Forte, and Yukiko Kenmochi. "Morphological hierarchies: A unifying framework with new trees". In: *Journal of Mathematical Imaging and Vision* 65.5 (2023), pp. 718–753. DOI: 10.1007/s10851-023-01154-x. URL: https://hal.science/hal-04137677 (cit. on p. 46).

[Pat18] Amit Patel. "Generalized persistence diagrams". In: *Journal of Applied and Computational Topology* 1.3–4 (May 2018), pp. 397–419. ISSN: 2367-1734. DOI: 10.1007/s41468-018-0012-6. URL: http://dx.doi.org/10.1007/s41468-018-0012-6 (cit. on pp. 47, 48).

[Pee+18] Jiju Peethambaran, Amal Parakkat, A. Tagliasacchi, et al. "Incremental Labelling of Voronoi Vertices for Shape Reconstruction: Curve Reconstruction". In: *Computer Graphics Forum* 38 (Oct. 2018). DOI: 10.1111/cgf.13589 (cit. on p. 183).

[Pel+06] Samuel Peltier, Sylvie Alayrangues, Laurent Fuchs, et al. "Computation of homology groups and generators". In: *Computers & Graphics* 30.1 (2006), pp. 62–69. ISSN: 0097-8493. DOI: https://doi.org/10.1016/j.cag.2005.10.011 (cit. on pp. 42, 43, 62, 84).

[PW24] Tuyen Pham and Hubert Wagner. *Computing Representatives of Persistent Homology Generators with a Double Twist*. 2024. arXiv: 2403.04100 [math.AT]. URL: https://arxiv.org/abs/2403.04100 (cit. on p. 84).

[PLX22] Chi Seng Pun, Si Xian Lee, and Kelin Xia. "Persistent-homology-based machine learning: a survey and a comparative study". In: *Artificial Intelligence Review* 55.7 (2022), pp. 5169–5213 (cit. on p. 46).

[RM09] Pedro Real and Helena Molina-Abril. "Cell AT-models for digital volumes". In: *International Workshop on Graph-Based Representations in Pattern Recognition*. Springer. 2009, pp. 314–323 (cit. on p. 44).

[Reb+19] Daniel Rebain, Baptiste Angles, Julien Valentin, et al. "LSMAT least squares medial axis transform". In: *Computer Graphics Forum*. Vol. 38. 6. Wiley Online Library. 2019, pp. 5–18 (cit. on p. 197).

[Rei16] Alexander Reich. "Minimum strictly fundamental cycle bases of planar graphs are hard to find". In: *Discrete Applied Mathematics* 205 (2016), pp. 150–159. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2015.12.001 (cit. on p. 167).

[Rob99] Vanessa Robins. "Towards computing homology from finite approximations". In: *Topology proceedings*. Vol. 24. 1. 1999, pp. 503–532 (cit. on p. 45).

[RS86] Julio Rubio and Francis Sergeraert. "1. Homologie effective". fre. In: *Cours de l'institut Fourier* 20.17 (1986), pp. 15–38. URL: http://eudml.org/doc/272555 (cit. on pp. 41, 43, 116).

[RS02] Julio Rubio and Francis Sergeraert. "Constructive algebraic topology". In: *Bulletin des Sciences Mathématiques* 126.5 (2002), pp. 389–412 (cit. on p. 40).

[Sel13] Peter H Sellers. *Combinatorial complexes: a mathematical theory of algorithms*. Springer Science & Business Media, 2013 (cit. on p. 32).

[Sie96] Dirk Siersma. "Voronoi Diagrams and Morse Theory of the Distance Function". In: (Sept. 1996) (cit. on pp. 194, 196, 203).

[SMV11] Vin de Silva, Dmitriy Morozov, and Mikael Vejdemo-Johansson. "Dualities in persistent (co)homology". In: *Inverse Problems* 27.12 (Nov. 2011), p. 124003. DOI: 10.1088/0266-5611/27/12/124003. URL: https://doi.org/10.1088/0266-5611/27/12/124003 (cit. on p. 91).

[SYM25] Anna SONG, Ka Man YIM, and Anthea MONOD. "Generalized Morse theory of distance functions to surfaces for persistent homology". In: *Advances in Applied Mathematics* 166 (2025), p. 102857 (cit. on pp. 180, 183, 204, 205).

[Ste08] Ernst STEINITZ. "Beiträge zur Analysis situs". In: *aus den Sitzungsberichten der Berliner Mathematischen Gesellschaft* 7 (1908), pp. 15–38 (cit. on p. 162).

[Tag+16] Andrea TAGLIASACCHI, Thomas DELAME, Michela SPAGNUOLO, et al. "3D Skeletons: A State-of-the-Art Report". In: *Computer Graphics Forum* 35 (May 2016). DOI: 10.1111/cgf.12865 (cit. on pp. 184, 196).

[The25] THE CGAL PROJECT. *CGAL User and Reference Manual*. 6.0.2. CGAL Editorial Board, 2025. URL: https://doc.cgal.org/6.0.2/Manual/packages.html (cit. on pp. 141, 198).

[Tie08] Heinrich TIETZE. "Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten". In: *Monatshefte für Mathematik und Physik* 19 (1908), pp. 1–118. URL: https://api.semanticscholar.org/CorpusID:120998023 (cit. on p. 162).

[Van98] Lou VAN DEN DRIES. *Tame topology and o-minimal structures*. Vol. 248. Cambridge university press, 1998 (cit. on p. 185).

[Wan+24] Ningna WANG, Hui HUANG, Shibo SONG, et al. "MATTopo: Topology-preserving medial axis transform with restricted power diagram". In: *arXiv preprint arXiv:2403.18761* (2024) (cit. on pp. 197, 207).

[WCK14] Adam C WILKERSON, Harish CHINTAKUNTA, and Hamid KRIM. "Computing persistent features in big data: A distributed dimension reduction approach". In: *2014 IEEE international conference on acoustics, speech and signal processing (ICASSP)*. IEEE. 2014, pp. 11–15 (cit. on p. 48).

[Woo+04] Zoë J. WOOD, Hugues HOPPE, Mathieu DESBRUN, et al. "Removing excess topology from isosurfaces". In: *ACM Trans. Graph.* 23.2 (2004), pp. 190–208. DOI: 10.1145/990002.990007 (cit. on p. 139).

[Wu+17] Pengxiang WU, Chao CHEN, Yusu WANG, et al. "Optimal Topological Cycles and Their Application in Cardiac Trabeculae Restoration". In: May 2017, pp. 80–92. ISBN: 978-3-319-59049-3. DOI: 10.1007/978-3-319-59050-9_7 (cit. on p. 93).

[YLJ18] Yajie YAN, David LETSCHER, and Tao JU. "Voxel cores: Efficient, robust, and provably good approximation of 3d medial axes". In: *ACM Transactions on Graphics (TOG)* 37.4 (2018), pp. 1–13 (cit. on p. 197).

[Yoo+16] Jaejun YOO, Eun Young KIM, Yong Min AHN, et al. "Topological persistence vineyard for dynamic functional brain connectivity during resting and gaming stages". In: *Journal of Neuroscience Methods* 267 (2016), pp. 1–13. ISSN: 0165-0270. DOI: https://doi.org/10.1016/j.jneumeth.2016.04.001 (cit. on pp. 98, 113).

[ZJ16] Qingnan ZHOU and Alec JACOBSON. "Thingi10K: A Dataset of 10,000 3D-Printing Models". In: *arXiv preprint arXiv:1605.04797* (2016) (cit. on p. 198).

[ZC05] Afra ZOMORODIAN and Gunnar CARLSSON. "Computing Persistent Homology". In: *Discrete and Computational Geometry* 33 (Feb. 2005), pp. 249–274. DOI: 10.1007/s00454-004-1146-y (cit. on pp. 46, 48).

ANNEXES

A. Supplementary proofs

Lemma (Lemma 1.6.13)**.** *Let $X = (P,S,C)$ be a HDVF for (K,∂) with reduction (f,g,h). Suppose that there is $C^* \subseteq C_q$ such that for all $\gamma \in C^*$, $g(\gamma)$ is a cycle, and such that $([g_q(\gamma)]^K)_{\gamma\in C^*}$ is linearly independent in $H_q(K)$.*
Then, there exists $X' = (P',S',C')$ a perfect HDVF for (K,∂) such that $X \prec X'$ and $C^* \subseteq C'$.

Proof. [*Proof of Lemma 1.6.13*] Using the completion Corollary 1.6.11, there exists $X'' = (P'',S'',C'')$ a perfect HDVF for (K',∂) with reduction (f'',g'',h'') such that $P \subseteq P''$ and $S \subseteq S''$, but a priori C^* is not included in C''.

The idea of the proof is to first show the injectivity of $\mathtt{j}_{C''\setminus C^*} \circ f'' \circ \mathtt{i}_{P''\cap C^*}$ and then use it to find a set $\Gamma \subseteq C''\setminus C^*$ such that $X' := \mathtt{M}^{\Gamma}_{P''\cap C^*} X''$ is a valid operation.

1. First, we show that $f'' \circ \mathtt{i}_{C^*}$ is injective:
   Suppose that there is $c \in \operatorname{Span} C^*$ such that $f''(c) = 0$. By Lemma 1.6.3, we write $g(c) = c + s$ with $s \in \operatorname{Span} S \subseteq \operatorname{Span} S''$. With Definition 1.4.2-5. we have:

$$\begin{aligned} g(c) &= g''f''g(c) + \partial h''g(c) + h''\partial g(c) \\ &= g''f''(c+s) + \partial h''g(c) && \text{because } \partial g = 0 \\ &= g''f''(c) + g''f''(s) + \partial h''g(c) \\ &= g''f''(c) + \partial h''g(c) && \text{as } f''\circ \mathtt{i}_{S''} = 0 \text{ and } s \in \operatorname{Span} S'' \\ &= \partial h''g(c) \in \partial(K') && \text{because } f''(c) = 0 \end{aligned}$$

   Hence, $g(c)$ is a boundary in K' so $[g(c)]^{K'} = 0$. By assumption, $\{[g_q(\gamma)]^{K'}\}_{\gamma\in C^*}$ is linearly independent, and $c \in \operatorname{Span} C^*$, therefore we obtain that $c = 0$.

2. Second, we show the injectivity of $\mathtt{j}_{C''\setminus C^*} \circ f'' \circ \mathtt{i}_{P''\cap C^*}$:
   Suppose that there is $c \in \operatorname{Span}(P''\cap C^*)$ such that $\mathtt{j}_{C''\setminus C^*} f''(c) = 0$. As $\operatorname{im} f'' \subseteq \operatorname{Span} C''$, this implies that $f''(c) \in \operatorname{Span}(C''\cap C^*)$.

   As $f''\circ \mathtt{i}_{C''} = id_{C''}$, we obtain that $f''(f''(c)) = f''(c)$. As a consequence, we have $f''(c - f''(c)) = 0$ which implies $c = f''(c)$ by injectivity of $f''\circ \mathtt{i}_{C^*}$ (1.). However, we have $c \in \operatorname{Span}(P''\cap C^*)$ and $f''(c) \in \operatorname{Span}(C''\cap C^*)$, so $c = 0$ (because $P''\cap C'' = \emptyset$).

3. Third, we show that $C^* = (P''\cap C^*) \sqcup (C''\cap C^*)$:
   By contradiction suppose $\gamma \in S''\cap C^*$. We have $g(\gamma) \in \gamma + \operatorname{Span} S$ by definition of g for a HDVF (Proposition 1.6.1), hence we obtain $g(\gamma) \in \gamma + \operatorname{Span} S''$ as $S \subseteq S''$. Moreover, $g(\gamma)$ is a cycle (by assumption) so we have $g(\gamma) \in \ker\partial \cap (\gamma + \operatorname{Span} S'')$.

   Hence, as $\gamma \in S''$, $g(\gamma)$ is a cycle only composed of cells in S'', which implies that $g(\gamma) = 0$ by Lemma 1.6.3. By injectivity of g, we obtain $\gamma = 0$ which is impossible.

As a result, $S'' \cap C^* = \emptyset$ and we obtain $C^* = (C^* \cap P'') \sqcup (C^* \cap C'')$ because (P'', S'', C'') is a partition.

Finally, we can define the suited HDVF X':
As $\mathtt{j}_{C''\setminus C^*} \circ f'' \circ \mathtt{i}_{P''\cap C^*}$ is injective (2.), it admits a invertible sub-matrix of maximal rank. Hence, there exists a subset $\Gamma \subseteq C''\setminus C^*$ such that $\mathtt{j}_\Gamma \circ \mathtt{j}_{C''\setminus C^*} f'' \mathtt{i}_{P''\cap C^*}$ is an invertible square matrix. Therefore, $\mathtt{j}_\Gamma \circ f'' \circ \mathtt{i}_{P''\cap C^* C^*}$ is invertible, so by Definition 1.6.3, $\mathtt{M}^\Gamma_{P''\cap C^*}$ is a valid generalized operation for X''.

To conclude, denote $X' = (P', S', C') := \mathtt{M}^\Gamma_{P''\cap C^*} X''$, we have:

$$P' = (P'' \sqcup \Gamma)\setminus(P'' \cap C^*) \qquad S' = S'' \qquad C' = \big(C'' \sqcup (P'' \cap C^*)\big)\setminus\Gamma$$

- $P \subseteq P'$ because $P \subseteq P''$ and $P \cap (P'' \cap C^*) = \emptyset$.
- $S \subseteq S'$ because $S \subseteq S''$.
- $C^* \subseteq C'$ because $C^* = (C'' \cap C^*) \sqcup (P'' \cap C^*)$ (3.) and $\Gamma \cap C^* = \emptyset$. □

Proposition (Proposition 4.3.9)**.** *Every subdivision K' of a complex K can be decomposed as a sequence of elementary subdivisions: $K = K^0 \prec K^1 \prec \ldots \prec K^N = K'$.*

Proof. [*Proof of Proposition 4.3.9*] Let (K', ∂) be a subdivision of (K, ∂), with inclusion function i, χ defined as previously. We order the cells of K by increasing dimension: $\tau_1, \tau_2, \ldots \tau_N$. Denote $T_j := i^{-1}(\tau_j) \subseteq \overline{K'}$.

Let $j \in [0, N]$. We define $\overline{K^j} := T_1 \sqcup \cdots \sqcup T_j \sqcup \{\tau_{j+1}, \ldots, \tau_N\}$, and $K^j = \operatorname{Span} \overline{K^j}$. We have $K^0 = K$ and $\overline{K^N} = \overline{K'}$.

We denote $i_j : \overline{K^j} \to \overline{K^{j-1}}$ the cell inclusion and $\chi^j : K^{j-1} \to K^j$ the corresponding injective map, defined on cells and extended linearly:

$$i_j(\sigma) = \begin{cases} \tau_j & \text{if } \sigma \in T_j \\ \sigma & \text{otherwise} \end{cases} \qquad \chi^j(\sigma) = \begin{cases} \chi(\tau_j) & \text{if } \sigma = \tau_j \\ \sigma & \text{otherwise} \end{cases}$$

Hence, composing χ^ks from $k = 0$ to $k = j$ results in $\chi^{1\to j} := \chi^j \circ \cdots \circ \chi^1 : K \to K^j$:

$$\chi^{1\to j}(\tau_k) = \begin{cases} \chi(\tau_k) & \text{if } k \le j \\ \sigma & \text{if } k > j \end{cases}$$

We define $\partial^j : K^j \to K^j$, defined on cells and extended linearly:

$$\partial^j(\sigma) = \begin{cases} \chi^{1\to j} \circ \partial(\sigma) & \text{if } \sigma \in \{\tau_{j+1}, \ldots, \tau_N\} \\ \partial'(\sigma) & \text{if } \sigma \in T_1 \sqcup \cdots \sqcup T_j \end{cases}$$

We now prove by induction on j that (K^j, ∂^j) is a well defined complex that is a subdivision of $(K^{j-1}, \partial^{j-1})$. To do this, we need to prove that $\partial^j \circ \partial^j = 0$, $|i_j^{-1}(\sigma)| = |\sigma|$ and $\partial^j \chi^j = \chi^j \partial^{j-1}$.

Base case. $(K^0, \partial^0) = (K, \partial)$ is a well defined geometric complex by assumption.

Induction case. Suppose that $\partial^k \circ \partial^k = 0$, $|i_k^{-1}(\sigma)| = |\sigma|$ and $\partial^k \chi^k = \chi^k \partial^{k-1}$ for all $k < j$.

- First we have easily $|i_j^{-1}(\sigma)| = |\sigma|$. Indeed, $|i_j^{-1}(\tau_j)| = |T_j| = |i^{-1}(\tau_j)| = |\tau_j|$. Otherwise if $\sigma \neq \tau_j$, $|i_j^{-1}(\sigma)| = |\{\sigma\}| = |\sigma|$.

- Second, we prove $\partial^j \circ \partial^j = 0$. Let $\sigma \in \overline{K^j}$. First case: $\sigma \in T_1 \sqcup \cdots \sqcup T_j$, then $\partial^j(\sigma) = \partial'(\sigma)$. As the cells are ordered by increasing dimension, the cells of $\partial'(\sigma)$ are also in $T_1 \sqcup \cdots \sqcup T_j$. As a result, $\partial^j(\partial'(\sigma)) = \partial'\partial'(\sigma) = 0$.

  Second case: $\sigma \in \{\tau_{j+1}, \ldots, \tau_N\}$, then $\partial^j(\sigma) = \chi^{1\to j}\partial(\sigma)$. Let us decompose $\partial(\sigma) = x + y$, with $x \in \operatorname{Span}\{\tau_1, \ldots, \tau_j\}$ and $y \in \operatorname{Span}\{\tau_{j+1}, \ldots, \tau_N\}$. We have $\chi^{1\to j}\partial(\sigma) = \chi^{1\to j}(x) + \chi^{1\to j}(y) = \chi(x) + y$, with $\chi(x) \in \operatorname{Span} T_1 \sqcup \cdots \sqcup T_j$. Hence, we get:

$$\begin{aligned}\partial^j\partial^j(\sigma) &= \partial^j\chi(x) + \partial^j(y) = \partial'\chi(x) + \chi^{1\to j}\partial(y)\\ &= \chi\partial(x) + \chi^{1\to j}\partial(y) && \text{as } \chi\partial = \partial'\chi\end{aligned}$$

  $x \in \operatorname{Span}\{\tau_1, \ldots, \tau_j\}$ so $\partial(x) \in \operatorname{Span}\{\tau_1, \ldots, \tau_j\}$ because the cells are ordered by increasing dimension. Hence, $\chi\partial(x) = \chi^{1\to j}\partial(x)$. As a result, $\partial^j\partial^j(\sigma) = \chi^{1\to j}\partial(x+y) = \chi^{1\to j}\partial\partial(\sigma) = 0$.

- Finally, we prove $\partial^j\chi^j = \chi^j\partial^{j-1}$. First case: $\sigma \in T_1 \sqcup \cdots \sqcup T_{j-1}$, then $\partial^{j-1}(\sigma) = \partial'(\sigma)$. As the cells are ordered by increasing dimension, the cells of $\partial'(\sigma)$ are also in $T_1 \sqcup \cdots \sqcup T_{j-1}$, so $\langle \partial'(\sigma), \tau_j \rangle = 0$. Hence,

$$\begin{aligned}\chi^j\partial^{j-1}(\sigma) &= \partial'(\sigma)\\ &= \partial^j(\sigma) && \text{as } \sigma \in T_1 \sqcup \cdots \sqcup T_j\\ &= \partial^j\chi^j(\sigma) && \text{as } \sigma \neq \tau_j\end{aligned}$$

  Second case: $\sigma = \tau_j$. As the cells are ordered by increasing dimension, the cells of $\partial^{j-1}(\tau_j)$ are in $T_1 \sqcup \cdots \sqcup T_{j-1}$. Moreover, the cells of $\partial(\tau_j)$ are in $\{\tau_1, \ldots, \tau_{j-1}\}$. Hence,

$$\begin{aligned}\chi^j\partial^{j-1}(\tau_j) &= \partial^{j-1}(\tau_j)\\ &= \chi^{1\to j-1}\partial(\tau_j) && \text{as } \tau_j \in \{\tau_j, \ldots, \tau_N\}\\ &= \chi\partial(\tau_j) && \text{as } \partial(\tau_j) \in \operatorname{Span}\{\tau_1, \ldots, \tau_{j-1}\}\\ &= \partial'\chi(\tau_j) && \text{as } \chi\partial = \partial'\chi\\ &= \partial'\chi^j(\tau_j)\\ &= \partial^j\chi^j(\tau_j) && \text{as } \chi^j(\tau_j) \in \operatorname{Span} T_1 \sqcup \cdots \sqcup T_j\end{aligned}$$

Third case: $\sigma \in \{\tau_{j+1}, \dots, \tau_N\}$, then $\partial^j(\sigma) = \chi^{1\to j}\partial(\sigma)$. Hence,

$$\begin{aligned}
\partial^j \chi^j(\sigma) &= \partial^j(\sigma) = \chi^{1\to j}\partial(\sigma) \\
&= \chi^j \circ \chi^{j-1} \dots \chi^1 \partial(\sigma) \\
&= \chi^j \circ \partial^{j-1}\chi^{j-1} \dots \chi^1(\sigma) && \text{as } \forall k < j,\ \partial^k \chi^k = \chi^k \partial^{k-1} \\
&= \chi^j \partial^{j-1}(\sigma) && \text{as } \sigma \in \{\tau_j, \dots, \tau_N\}\ \square
\end{aligned}$$

B. Gallery



Figure B.1 – Folding a dodecagon into a tri-torus. Acrylic painting presented during the *Illustration as a Mathematical Research Technique* workshop at IHP in Paris, in March 2026.

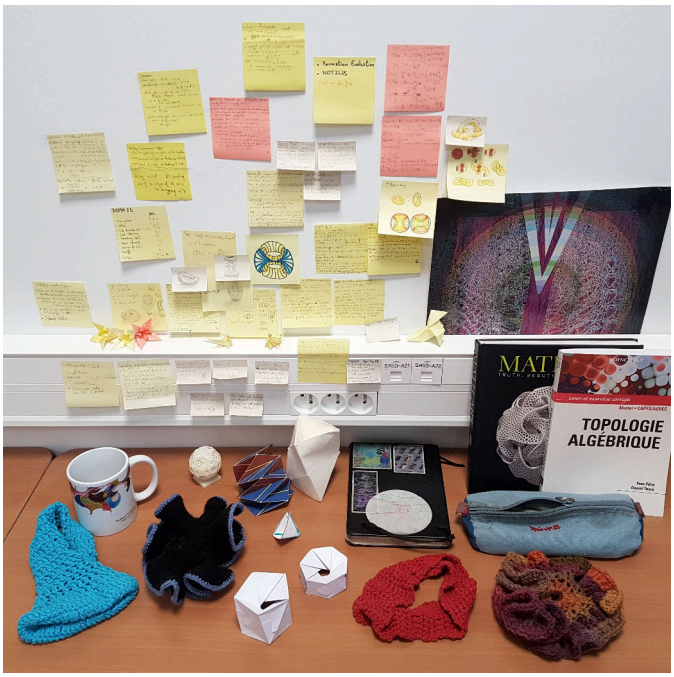


Figure B.2 – PhD desk: proofs on post-its and mathematic trinkets.

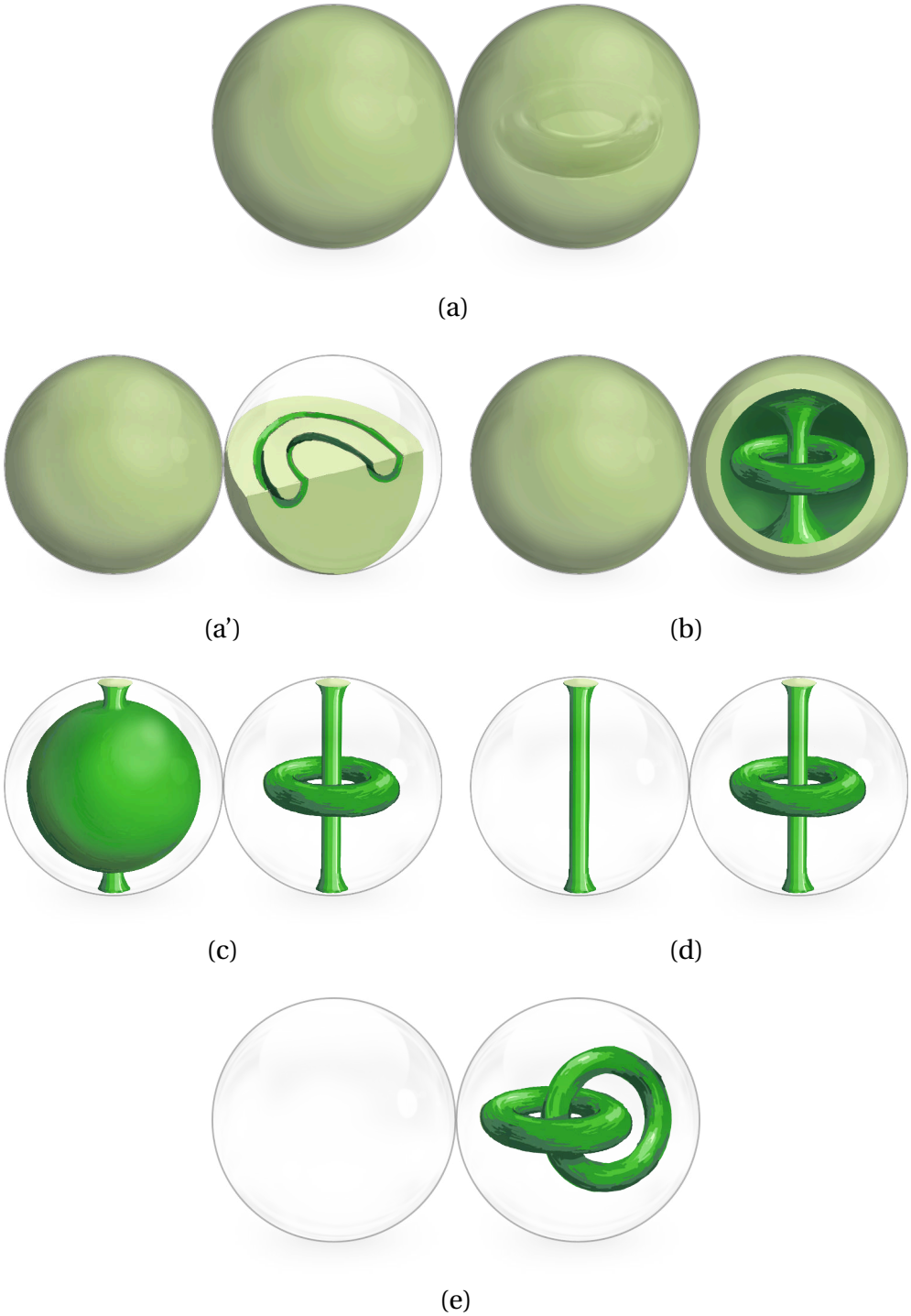


Figure B.3 – Continuous deformation of the complement of a torus in $\mathbb{S}^3$ (a) into two intertwined annuli (e), illustrating Alexander duality. $\mathbb{S}^3$ is represented with two 3-balls (hemispheres) glued together along their spherical boundary. This representation is known as the Gott projection. (a') shows a cross-sectional view of (a). (b) also includes a cross-sectional view, for clarity.



Figure B.4 – A scan of the PhD companion notebook.